Interaction of solitary waves in granular confined alignments and the quasi-equilibrium state

Edgar Ávalos
Graduate Institute of Biophysics and Center for Complex Systems,
National Central University, Chung-Li, Taiwan 320, Republic of China

Robert L. Doney
US Army Research Labs, Aberdeen Proving Grounds, MD 21005, USA

Surajit Sen
Department of Physics, State University of New York, Buffalo, New York, 14260-1500,USA

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We consider a chain of elastic beads, in which the individual grains repel upon contact according to the non-linear Hertz potential. We further assume that the chain is under zero loading, i.e., the grains barely touch one another. We show via careful numerical solution of the equations of motion that an impulse propagates as a solitary wave and that the collision of identical solitary waves propagating in opposite directions along the chain spawns a hierarchy of multiple weak solitary waves. We mention that the generated baby waves eventually must lead to a "volatile" equilibrium-like state, which we refer to as the quasi-equilibrium state.

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I. INTRODUCTION

The interaction of mechanical perturbations propagating in granular media exhibits striking nonlinear phenomena. When a mechanical pulse is initiated in a one-dimensional alignment of elastic grains, it propagates as a solitary wave [1]. Solitary waves are tight bundles of energy which can travel without dispersion in space or in time [2].

We assume that our beads are elastic objects. In a non-dissipative system, the solitary waves are expected to last forever. We consider such non-dissipative systems in this study. However when a mechanical perturbation travels from one grain to the next one, the geometry of the contact surfaces between adjacent grains changes and as per Hertz law, leads to a non-linear interaction between them. When grains get compress, mechanical energy can flow from one grain to the next. Hertz showed that the potential energy cost associated with the grain deformation is initially softer than harmonic. However, with increasing compression, the potential energy cost rises sharply. In order to minimize this potential energy cost in transferring energy from one grain to the next, the grain grain contact rapidly ends, thereby transferring an energy bundle, which in turn behaves as a solitary wave, These solitary waves travel in speeds that are proportional to their amplitude, with those carrying higher energy and involving larger amplitude compressions traveling faster than those with smaller amplitudes and causing less compression.

In 1985 Nasterenko reported the first time experimental evidence of solitary waves supported by granular alignments [3] and extensive experimental work was done by C. Coste *et al.* [4] and by S. Job, *et al.* [5]. Since then there has been special interest in the propagation of and the interaction of between the solitary waves.

In general it turns out that in a granular alignment under the influence of external forces [6, 7], the interaction between grains has both linear and non-linear components [2, 8]. When the beads in the linear array are non pre-compressed, the underlying interaction is purely non-linear and as a result the linear force is absent and hence sound propagation becomes impossible. These systems have hence been referred to by Nesterenko as 'sonic vacuum' systems.

During the last few years there has been increasing interest in the investigation of the properties of propagating waves in un-compressed discrete media. This interest is driven mainly by the existence of a new kind of solitary wave in these systems and by the ways in which these solitary waves interact with one another and with the system boundaries. These phenomena include the formation of secondary solitary waves [9] and the establishment of an equilibrium-like state due to the interactve nature of the of solitary waves [10]. Regarding the formation of secondary solitary waves, the most challenging efforts have been made to study the properties of these waves in different conditions as for instance, in granular chains with parity differences in the number of grains [11], or with non-uniform distribution of bead-size [12] or with different boundaries [5].

The second line of research deals with a novel collective state in which particle velocities satisfy a Gaussian distribution but the energy is not equi-partitioned in the system [13].

In this paper we briefly review the problem of the interaction of solitary waves in granular arrays with either themselves or with boundaries and introduce the basic features of the newly discovered quasi-equilibrium state. In both cases the original solitary waves are destroyed.

FIG. 1: Schematic setup of a granular chain at initial time. Two incoming pulses of equal velocity start simultaneously at the ends of the chain.

We focus mainly in the formation process of secondary solitary waves and the role of boundaries on their properties.

The Fig.1 shows a schematic setup of the numerical experiment of the collision of two identical solitary waves that start off at the ends of a monodisperse chain made of N grains and propagate through the system at equal speeds.

This paper is organized as follows: first we describe the numerical procedure used in our simulations to solve directly the equations of motion for all the grains in a granular chain. Next, we discuss the propagation and the interaction of solitary waves and discuss the formation of secondary solitary waves. We conclude by describing the basic properties of the eventual quasi-equilibrium state of the system.

II. THE MODEL AND SIMULATION DETAILS

We consider a linear arrangement of N identical spherical grains of radius R and mass m, interacting through a Hertzian contact force without pre-compresion. According to Hertz law, any two adjacent pair of beads labeled as i and $i+1$ interact as follows,

$$V(\delta_{i,i+1}) = \begin{cases} a\ \delta^n_{i,i+1}, & \delta > 0, \\ 0, & \delta \leqslant 0, \end{cases} \tag{1}$$

where n is 5/2 for spherical grains [8], $a = \frac{2}{5D}\sqrt{\frac{R}{2}}$, where $D = 3(1-\sigma^2)2Y$, is a constant which depends on the Young's modulus, Y, and the Poisson's ratio σ of the grains and $\delta_{i,i+1} \equiv 2R - (z_{i+1} - z_i)$, where z_i denotes the displacement of grain i from the equilibrium position. In granular systems, n depends upon the nature of the contact and $n > 2$. Eq. (1) is valid for grains made of an elastic material. However, the resulting interaction is non-linear due to the geometrical properties of the grains.

The equation of motion for the position of the i-th grain in the chain can be written as,

$$m\ddot{z}_i = na\left(\delta^{n-1}_{i,\ i-1} - \delta^{n-1}_{i+1,\ i}\right). \tag{2}$$

Formal solutions to Eq. (1) were attempted by Nesterenko [14] and later by S. Sen and M. Manciu [15]. Here we describe the creation of secondary solitary waves by the collision of two solitary waves. The creation of secondary solitary waves by collision of a solitary wave against a wall is described in Ref. [5]. We start with two identical impulses directed into the chain at the initial time, $t = 0$, by assigning $v_1(0) = -v_0$ and $v_N(0) = v_0$, with $v_0 > 0$, at the two ends of a chain made of N identical grains. The other grains are at rest at the initial time. These impulses lead eventually to the formation of identical solitary waves that are initiated at the ends of the chain. The initial impulse is all kinetic energy. It takes the system some space and time to convert this kinetic energy pulse into a solitary wave. Studies have shown that for spherical beads, where $n = 5/2$, the system takes about 10 grain diameters to form a solitary wave. The waves are about 7 grain diameters wide.

The system dynamics is obtained by direct time integration of the coupled Newtonian equations of motion via the 3^{rd} order Gear algorithm [16]. We choose 10^{-5} m, 2.36×10^{-5} kg and 1.0102×10^{-3} s as the units of distance, mass and time, respectively. The integration time step, dt, used was chosen small enough to resolve the finest details. For $n = 2.5$, i.e., spheres, we use $dt = 5\times10^{-7}$. The grain diameter is set to 100, i.e., 1 mm, $2R = 1$ and $a = 5657$ [17], a value ($= 4.14\times10^7$ N/m$^{3/2}$)which is the range of elasticity of silicate materials. In the cases here, we use an even chain made of $N = 500$ and $v_0 = 5\times10^{-4}$.

III. FORMATION OF SECONDARY SOLITARY WAVES

By using the method described above, we perform extensive computations to simulate the collision of solitary waves. Once each pulse is initiated at each end of the chain, these propagate as solitary waves of 7 grain diameters width. This size is independent of both, amplitude and grain size [18]. The Fig. 2 shows the velocity profiles of the two solitary waves in incoming trajectories prior to the collision. This figure shows also that each solitary wave has a constant width. Another important feature is that their propagation velocity scales with amplitude by an exponent 1/5. This is consistent with results obtained by Nesterenko [14] and by Sen and Manciu [15]. The solitary waves approach each other and cross at the edge of the two central grains in the chain with even N or at the center of the central grain in a chain with odd N. After the crossing occurs, the solitary waves continue until they reach the opposite ends of the chain.

Now we turn our attention on the detailed dynamics of the crossing event, which leads to the formation of the so-called secondary solitary waves. During the collision the original solitary waves do not cancel each other at the intersection point. In general, during the collision all beads in the chain move out from their original position. Only the central grain in an odd chain remains static at the time of collision. In the even chain all the participating grains, including the one in the region where the crossing occurs, move out from their equilibrium positions during the complete interval of time. Therefore, immediately after the collision occurs, there is more kinetic energy

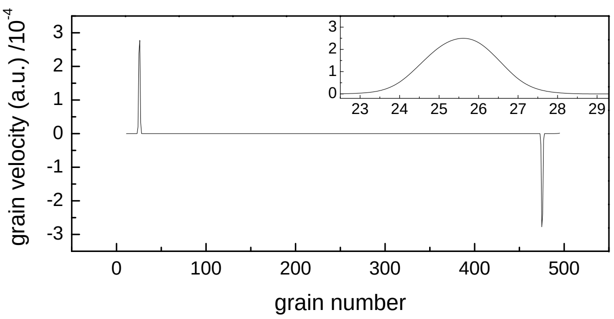

FIG. 2: Snapshot of the two solitary waves in an early stage traveling in incoming trejectories. (a.u.) on the y-axis stands for arbitrary units. The inset shows in detail one of the pulses with a constant width of 7 diameters.

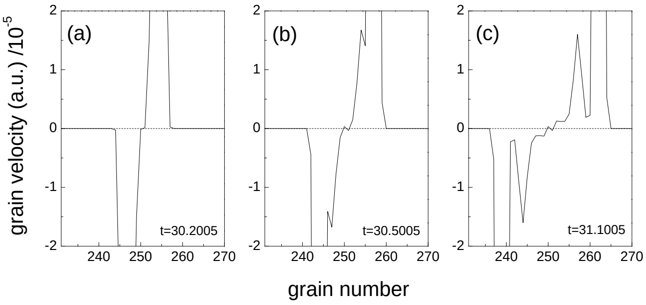

FIG. 3: Formation of secondary solitary waves. The central region in the y-axis has been amplified 10 times to show the finest detail of the formation. (a.u.) on the y-axis stands for arbitrary units.

available in the even case.

During the collision solitary waves break down and reform; which means that during a short interval of time the profiles of the solitary waves lose their initial shape and are reformed again to their original shape with secondary solitary waves as a by-product [5]. In Fig. 3 we can see the very precise moment when secondary solitary waves are formed.

After the crossing occurs, we carefully examine the region behind the crossed solitary waves. Using higher resolution to observe the kinetic energies of the individual grains, we find that multiple solitary waves of progressively decreasing amplitudes form in the collision region after the collision. The less energetic the solitary wave is, the slower it moves. These waves are called secondary solitary waves. In Fig. 4 we see the secondary solitary waves created as a by-product after the collision. It is worth observing that the system cannot transport energy via sound waves in this ‘sonic vacuum’ system. Thus, all energy must eventually be bundled into solitary waves. Given that these waves are 7 grain diameters wide, it is easy to see that at least 7 grain collisions would be needed for these waves to be born. Thus, the time interval in which the waves are created seem not to be according to any simple pattern. These waves are several order of magnitude smaller than the original ones. Secondary solitary waves are formed in pairs of the same amplitude but move away from each other. Subsequently more secondary solitary waves are formed with decreasing amplitudes. Depending on the collision region, secondary solitary waves can be up to three orders of magnitude less energetic than the original solitary waves [11]. The amplitudes of secondary solitary waves depend on the collision region and also on the boundaries.

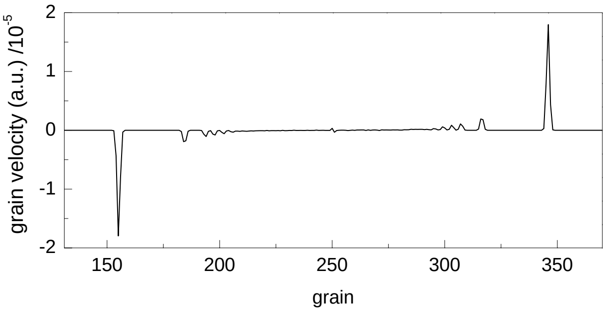

FIG. 4: Secondary solitary waves created after the collision of two solitary waves in a granular chain made of an even number of beads. (a.u.) on the y-axis stands for arbitrary units.

The energy associated to secondary solitary waves depends not only on the on the collision region but also on the softness of the inter-grain interaction. Steeper potentials produce more secondary solitary waves.

When the collision happens on a wall, again secondary solitary waves are created after the collision and in this case, their amplitude can be controlled by tuning the Yung’s modulus of the wall; a softer wall promotes stronger secondary solitary waves than harder ones [5].

IV. CONCLUSIONS

In this paper we gave a description of the propagation of solitary waves in a chain made of spherical beads. We shown that when two solitary waves collides one against the other or one against a wall, a hierarchy of multiple weak solitary waves is formed as a by-product. The amplitude of these secondary waves depends on the parity of the number of beds, which indicates that granularity plays a critical role on the emerging properties. In the case of wall-collison, the softness of the wall determines the intensity of the resulting secondary waves.

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