Problem 1

A marble of mass M is on a track which consists of a hill leading into a loop. The loop has radius R and the hill has height H and H > 2R.

a) What is the marble's speed at the bottom of the loop?

b) At the top of the loop?

c) What is the force of the track on the marble at the top of the loop?

d) What is the minimum value of H so the marble can complete the loop without falling?

Solution 1

a)Use conservation of energy.

$$E = \frac{1}{2}mv^2 + Potential$$

At the top of the hill Potential = mgH and v = 0.

At the bottom Potential = 0 and v = ?.

$$E_{top} = mgH$$

$$E_{bottom} = \frac{1}{2}mv^2$$

Conservation of energy says that $E_{top} = E_{bottom}$

$$mgH = \frac{1}{2}mv^2$$

$$v = \sqrt{2gH}$$

b) Same procedure

$$E_{initial} = mgH$$

$$E_{final} = \frac{1}{2}mv^2 + mg2R$$

$$mgH = \frac{1}{2}mv^2 + \mathrm{mg2R}$$

$$v = \sqrt{2gH - 4gR}$$

c) Use Newton's second law

$$\sum F = ma$$

The forces on the car are the normal force which point perpendicular to the track, and gravity which points straight down. The acceleration is the centripetal acceleration given by $a = -\frac{v^2}{R}$

$$-mg - N = -m\frac{v^2}{R}$$

$$N = m\frac{v^2}{R} - mg$$

Plug in the speed found in part b

$$N = m\frac{2gH - 4gR}{R} - mg = mg\left(\frac{2H}{R} - 5\right)$$

d) The marble will fall if the normal force is zero.

$$N = mg\left(\frac{2H}{R} - 5\right) = 0$$

$$\frac{2H}{R} = 5$$

$$\frac{5}{2}R = H$$

Problem 2

A potential energy is given as

$$U(r) = U_0\left(\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right)$$

The potential can be thought of as representing the forces between two atoms, where r is the separation distance. Find the distance r at which the force between the atoms is zero. What is the potential at this distance?

Solution 2

Force in terms of potential is given by

$$\vec{F} = -\frac{dU}{dr}\hat{r}$$

We are interested in the case where F = 0.

$$\vec{F} = -\frac{dU}{dr}\hat{r} = -\frac{d}{dr}U_0\left(\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^6\right)\hat{r} = U_0\left(\frac{-12r_0^{12}}{r^{13}} + \frac{12r_0^6}{r^7}\right)\hat{r} = 0$$

$$U_0\left(\frac{-12r_0^{12}}{r^{13}} + \frac{12r_0^6}{r^7}\right)\hat{r} = 0$$

Solve for r, the separation distance

$$\left(\frac{-12r_0^{12}}{r^{13}} + \frac{12r_0^6}{r^7}\right) = 0$$

$$\frac{12}{r^{13}}r_0^{12} = \frac{12}{r^7}r_0^6$$

$$r_0^6 = r^6$$

$$r = r_0$$

To find the potential at this distance plug in r_0 for r in the potential formula.

$$U(r_0) = U_0\left(\left(\frac{r_0}{r_0}\right)^{12} - 2\left(\frac{r_0}{r_0}\right)^6\right) = U_0(1-2) = -U_0$$

Problem 3

A simple pendulum of length, L = 1.1m and mass, m = 0.23Kg is released from the horizontal position. When the mass is at the lowest point it hits a nail located a distance h above the lowest point. The mass then loops around the nail. How large can h be so that the string remains taut even when the mass is directly above the nail.

Solution 3

If there is tension in the string then the string is taut. Use conservation of energy to find the velocity at the point when the mass is above the nail.

At the release point

$$E = mgL$$

At the point above the nail

$$E = mg2h + \frac{1}{2}mv^2$$

E is constant

$$mgL = mg2h + \frac{1}{2}mv^2$$

$$v = \sqrt{2g(L-2h)}$$

Use Newton's second law to find the tension

$$\sum F = ma$$

$$-mg - T = -m\frac{v^2}{R}$$

$$-mg - T = -m\frac{2g(L-2h)}{h}$$

$$T = mg(\frac{2L}{h} - 5)$$

The maximum of h is found from the minimum tension, T = 0.

$$0 = mg(\frac{2L}{h} - 5)$$

h = (2/5)L= 0.4m

Problem 4

The potential energy of a satellite in a circular orbit is $U(r) = -\frac{GMm}{r}$ where G is a constant, r is the radius of the orbit, M is the mass of the central planet and m is the mass of the satellite.

a) Find the force on the satellite.
b) Find the kinetic energy of a particle in this orbit.
c) Find the total energy of the orbit.

Solution 4

a) Force is given by the negative derivative of the potential.

$$\vec{F} = -\frac{dU}{dr}\hat{r}$$

$$\vec{F} = \frac{d}{dr}\frac{GMm}{r}\hat{r} = -\frac{GMm}{r^2}\hat{r}$$

b) Use Newton's second law

$$\sum \vec{F} = m\vec{a}$$

$$-\frac{GMm}{r^2}\hat{r} = -m\frac{v^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

The kinetic energy

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\frac{GM}{r}$$

c) The total energy

$$E = KE + PE$$

$$E = \frac{GMm}{2r} - \frac{GMm}{r} = -GMm/2r$$

Problem 5

A jump takes a car through a vertical distance H from the start to the take off point. The angle of take off , theta, can be set to any angle. Find the angle that gives the maximum range for a landing on a plane that sit a distance D below the take off point.

Solution 5

From conservation of energy find the take off velocity.

At the top of the slope

$$E = mgH$$

At the bottom of the slope, the take off point.

$$E = \frac{1}{2}mv^2$$

Energy is constant

$$mgH = \frac{1}{2}mv^2$$

$$v = \sqrt{2gH}$$

Now use the x and y position equations for a projectile

$$x_f = x_i + v_i t$$

$$x = (\sqrt{2gH}\cos\theta)t$$

$$y_f = y_i + v_i t - \frac{1}{2}gt^2$$

$$0 = D + (\sqrt{2gH}\sin\theta)t - \frac{1}{2}gt^2$$

Use the x equation to eliminate time from the y equation

$$0 = D + \frac{x\sqrt{2gH}\sin\theta}{\sqrt{2gH}\cos\theta} - \frac{1}{2}g\left(\frac{x}{\sqrt{2gH}\cos\theta}\right)^2$$

$$0 = -D - x\tan\theta + \frac{x^2}{4H\cos^2\theta}$$

Use the quadratic formula to solve for x

$$(\tan\theta \pm \sqrt{\tan^2\theta + 4\frac{1}{4H\cos^2\theta}D}) \div \frac{2}{4H\cos^2\theta}$$

$$2H\cos^2\theta(\tan\theta \pm \sqrt{\tan^2\theta + D/H\cos^2\theta})$$

$$x = 2H\cos\theta(\sin\theta \pm \sqrt{\sin^2\theta + D/H})$$

Now x can be maximized with respect to theta using dx/dθ = 0.

$$\frac{d}{d\theta} 2H\cos\theta(\sin\theta \pm \sqrt{\sin^2\theta + D/H}) = 0$$

$$\frac{dx}{d\theta} = 2H(-\sin\theta(\sin\theta \pm \sqrt{\sin^2\theta + D/H}) + \cos\theta(\cos\theta \pm \frac{\sin\theta\cos\theta}{\sqrt{\sin^2\theta + D/H}})) = 0$$

Which can be written

$$\sin^4\theta - \cos^4\theta = -(D/H)\sin^2\theta.$$

Or

$$(\sin^2\theta)^2 - (1-\sin^2\theta)^2 = -D/H\sin^2\theta$$

Call sin^2θ = z

$$z^2 - (1-z)^2 = -(D/H)z$$

$$z^2 - ((1-z)(1-z)) = -(D/H)z$$

$$z^2 - (1-2z+z^2) = -(D/H)z$$

$$-1+2z = -(D/H)z$$

$$z = \sin^2\theta = \frac{1}{2+D/H}$$

$$\theta = \arcsin\left(\frac{1}{\sqrt{2+D/H}}\right)$$