Problem 1

A canon is fired with a muzzle velocity of 330m/s from the origin. The cannon shell has a fuse which causes it to explode 42.0s after being fired. If the target moves according to the position vector

$$\vec{r} = 40t + 600\hat{i} + 6\hat{j} + 100\cos(2^\circ \frac{1}{s} t)\hat{k}$$

Find the initial angles, with respect to the x axis and z axis, from which the cannon should be fired.

Solution 1

The motion in each dimension can be treated separately. Find the initial velocities in each direction resolving the initial velocity vector into component form in 3D.

$$v_{0x} = |\vec{v}|\sin\theta\cos\phi$$

$$v_{0y} = |\vec{v}|\sin\theta\sin\phi$$

$$v_{0z} = |\vec{v}|\cos\theta$$

Where θ is the angle measured from the z axis and φ is the angle measured from the x – axis. The position equations for the shell are

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 = v_{0x}t = (|\vec{v}|\sin\theta\cos\phi)t$$

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = (|\vec{v}|\sin\theta\sin\phi)t$$

$$z = z_0 + v_{0z}t + \frac{1}{2}a_z t^2 = (|\vec{v}|\cos\theta)t - \frac{1}{2}gt^2$$

The position of the target is given as

$$x = 40\frac{m}{s}t + 600m$$

$$y = 6m$$

$$z = 100m\cos(2\frac{m}{s}t)$$

Each coordinate of the shell must equal each coordinate of the target at the time when the shell explodes.

$$(|\vec{v}|\sin\theta\cos\phi)(42s) = (40\frac{m}{s})(42s) + 600m$$

$$(|\vec{v}|\sin\theta\sin\phi)(42s) = 6m$$

$$(|\vec{v}|\cos\theta)(42s) - \frac{1}{2}9.8(42s)^2 = 100m\cos(42s * 2)$$

From the 3rd equation theta can be found.

$$\theta = \arccos\left(\frac{100\cos(84^\circ) + 4.9(42)^2}{(330)(42)}\right) = 51°$$

Now the using the x and y equations

$$\frac{|\vec{v}|\sin\theta\sin\phi(42s)}{|\vec{v}|\sin\theta\cos\phi(42s)} = \frac{(40\frac{m}{s})(42s)+600m}{6m}$$

$$\tan\phi = 380$$

$$\phi = 89.9^\circ$$

Problem 2

A gardener is watering his garden. He sees that his neighbor's house is on fire. If the gardener is a distance d away from the burning house and he directs the hose at an angle θ above the ground. The water leaving the hose has an initial velocity Vo. At what height, h, does the water strike the house? Assume that the initial height of the hose is 0m.

Solution 2

First find the initial velocity in each direction by resolving the initial velocity vector into its components.

$$v_{0x} = v_0\cos\theta$$

$$v_{oy} = v_0\sin\theta$$

Now use the kinematic equation for the motion on the x-axis to solve for the time the water takes from the hose to the house.

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

The water starts at 0, it experiences no acceleration in the x direction and its final position on the x-axis is d. The equation becomes

$$x = v_{0x}t$$

$$d = v_0\cos\theta t$$

$$t = \frac{d}{v_0\cos\theta}$$

Now that the time of the trip is known, use the y equation to find the height

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

$$h = v_{0y}t - \frac{1}{2}gt^2$$

$$h = v_0\sin\theta t - \frac{1}{2}gt^2$$

Plug in t

$$h = d\frac{v_0\sin\theta}{v_0\cos\theta} - \frac{1}{2}g\frac{d^2}{v_0^2\cos^2\theta}$$

$$h = d\tan\theta - \frac{1}{2}g\frac{d^2}{v_0^2\cos^2\theta}$$

Problem 3

A marble is slide down the table to a person. The person fails to catch the marble and it slides of the table and strikes the floor 2.4m from the edge of the table. If the height of the table is 0.790m what was the marble velocity as it left the table. What is the velocity vector when the marble strikes?

Solution 3

The marble as it leaves the table will only have velocity in the x direction.

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$

Initial position on the x axis is 0m and there is no acceleration in the x direction.

$$x = v_{0x}t$$

Solve for the time of the trip

$$t = \frac{x}{v_{0x}}$$

Now, use the y equation to find Vox

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$

$$y = h - \frac{1}{2}gt^2$$

$$y = h - \frac{1}{2}g(\frac{x}{v_{0x}})^2$$

$$v_{0x} = \sqrt{\frac{gx^2}{2h}}$$

Plug in the numbers, x = 0.790m, h = 2.4m, g = 9.8

$$v_{0x} = \sqrt{\frac{(9.8)(0.790)^2}{(2)(2.4)}} = 1.1m/s$$

To find the velocity vector the x and y components of the velocity are needed. There is no acceleration in the x – direction so the x – velocity remains constant. Vx = 1.1m.
The y – direction has acceleration –g.

$$v_f^2 = v_i^2 + 2a(y_f - y_i)$$

$$v_f^2 = 0 - 2g(0 - 0.790m) = 15.5\frac{m^2}{s^2}$$

Vy = -3.9m/s

The velocity vector at the time of impact is

$$\vec{v} = 1.1\frac{m}{s}\hat{i} - 3.9\frac{m}{s}\hat{j}$$

Problem 4

A canon fires a projectile at a muzzle velocity of Vo. Find an expression for the horizontal range in terms of the firing angle theta. Find the angle to get the longest range. What is the max height at this angle?

Solution 4

The horizontal range is given by the x equation

$$x = v_0 \cos\theta(t)$$

Use the y equation to eliminate time

$$0 = v_0 \sin\theta(t) - \frac{1}{2} g t^2$$

$$t = \frac{-v_o \sin\theta \pm \sqrt{v_0^2 \sin^2\theta}}{-g} = 0, \frac{2v_0 \sin\theta}{g}$$

$$x = v_0 \cos\theta \frac{2v_0 \sin\theta}{g}$$

To find the angle for maximum range take the theta derivative and set it equal to zero.

$$\frac{dx}{d\theta} = \frac{d}{d\theta} v_0 \cos\theta \frac{2v_0 \sin\theta}{g} = \frac{2v_0^2}{g}\cos^2\theta - \frac{2v_0^2}{g}\sin^2\theta = 0$$

Solve for theta

$$\cos^2\theta = \sin^2\theta$$

$$\tan\theta = 1$$

$$\theta = 45°$$

To find the maximum height at 45 degrees initial angle, use the y velocity equation

$$v_y = v_0 \sin\theta - gt$$

The projectile is launched upward, its velocity decreases until it has reached it max height where the velocity is momentarily zero. Use the velocity equation to find the time when the velocity goes to zero.

$$(v_0 \sin 45) / g = t$$

Just use the y position equation to find the height at this time.

$$y = v_0 \sin\theta(t) - \frac{1}{2} g t^2 = \frac{v_0^2 \sin^2 45}{g} - \frac{v_0^2 \sin^2 45}{2g} = \frac{v_0^2 \sin^2 45}{2g}$$

Problem 5

A particle is attached to a pole and moves in a circular path of radius R = 0.4m in a plane. The mass has a velocity, v = 18m/s. If the particle moves in the x-y coordinate system and at t=0s it is located at θ = 0°, where theta is the angle from the x-axis.

a) What is the position of the particle at t =0.1s

b) What is the acceleration vector at t = 0s

c) What is the acceleration vector at θ = 90°

Solution 5

a) From the speed of the particle it is possible to get the angle theta as a function of time. Speed is the distance traveled divided by the time.

$$|\vec{v}| = \frac{2\pi R}{t}$$

$$18m/s = \frac{2\pi(0.4m)}{t}$$

$$t = \frac{2\pi(0.4m)}{18m/s} = 0.1396s$$

Now get the angular velocity which is defined as the angle traversed over the time.

$$\omega = \frac{360^\circ}{t} = 2578.3^\circ / s$$

Angular velocity is also the derivative of the angle with respect to time. Integrating gives the angle as a function of time.

$$\frac{d\theta}{dt} = 2578.3^\circ / s$$

$$\int_0^\theta d\theta = \int_0^t 2578.3^\circ / s dt$$

$$\theta(t) = 2578.3^\circ t$$

Now the angle can be found at any time and from the angle the x and y coordinates can be resolved from the position vector.

$$x = R\cos\theta$$
$$y = R\sin\theta$$

At t = 0.1s

$$\theta(0.1s) = 2578.3^\circ (0.1s) = 257.8^\circ$$
$$x = 0.4\cos 257.8 = -0.08m$$
$$y = 0.4\sin 257.8 = -0.39m$$

b)c) For constant circular motion the acceleration vector points to the origin

$$\vec{a} = -\frac{v^2}{R}\hat{R} = -\frac{18^2}{0.4}\hat{R} = -810m/s^2\hat{R}$$

The magnitude is constant but the direction is changing. The vector can be written in component form.

$$\vec{a} = -\frac{v^2}{R}\hat{R} = -\frac{v^2}{R}(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}) = -810\frac{m}{s^2}(\cos(\theta)\hat{i} + \sin(\theta)\hat{j})$$

As a function of time from the time dependence of theta

$$\vec{a} = -810\frac{m}{s^2}(\cos(2578t)\hat{i} + \sin(2578t)\hat{j})$$

b) t = 0

$$\vec{a} = -810\frac{m}{s^2}(\cos(2578t)\hat{i} + \sin(2578t)\hat{j}) = -810\frac{m}{s^s}\hat{i}$$

c) θ = 90

$$\vec{a} = -810\frac{m}{s^2}(\cos(\theta)\hat{i} + \sin(\theta)\hat{j}) = -810\frac{m}{s^2}\hat{j}$$