Problem 1
A physical pendulum consists of a rigid body of some length with and axis of rotation through one end. For this problem the rigid body is a rod of mass m and length L. Write the equation of motion for the physical pendulum.
Solution 1
The moment of inertial of the rod

$$I = \int r^2 dm$$

$$I = \int_0^L x^2 \frac{m}{L} dx = \frac{m}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{m}{3L} L^3 = \frac{1}{3} mL^2$$

The torque from gravity

$$\vec{\tau} = \vec{r}_{cm} \times \vec{F} = (\frac{L}{2}\sin(\theta)\hat{i} - \frac{L}{2}\cos(\theta)\hat{j}) \times (-mg\hat{j}) = -\frac{L}{2}\sin(\theta) mg\hat{k}$$

Newton's second law for rotational motion

$$\sum \tau = I\alpha$$

$$-\frac{L}{2}\sin(\theta)mg = \frac{1}{3}mL^2\alpha$$

$$-\frac{L}{2}\sin(\theta)mg = \frac{1}{3}mL^2\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \frac{3g}{2L}\sin\theta = 0$$

Assume the pendulum oscillates through small angles
For small angles $\sin\theta \sim \theta$

$$\frac{d^2\theta}{dt^2} + \frac{3g}{2L}\theta = 0$$

This is the harmonic oscillator differential equation which has solutions of the form

$$\theta = A\sin(\sqrt{\frac{3g}{2L}}t + \phi)$$

where A and phi are constants which are determined from the initial conditions.

Problem 2
Two point masses, $m_1 = 2.0$kg and $m_2 = 7.2$kg, are connected by a light pole 4.0 meters long. They rotate about their center of mass with angular velocity $\omega=\pi$rad/s.
a) What is the moment of inertia?
b) The angular momentum?
c) The rotational axis is suddenly changed to ½ way between the masses what is the new angular velocity
Solution 2
The location of the center of mass must first be found.

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i}$$

where x is taken as the distance from m_1

$$x_{cm} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} = \frac{(0)(2.0m) + (4.0m)(7.2kg)}{2.0kg + 7.2kg} = 3.1m$$

Now the moment of inertia

$$I_{cm} = \sum_i r_i^2 m_i$$

where r is the distance from the center of mass

$$I_{cm} = (3.1m)^2 (2.0kg) + (0.87m)^2 (7.2kg) = 25kgm^2$$

b) Angular momentum

$$L = I\omega$$

$$L = (25kgm^2)(\pi rad/s) = 79kgm^2/s$$

c) Angular momentum is conserved

$$L = 79kgm^2/s = I\omega$$

The new moment of inertia is

$$I = \sum_i r_i^2 m_i = (2.0m)^2 (2.0kg) + (2.0m)^2 (7.2kg) = 37kgm^2$$

$$\frac{L}{I} = \omega = \frac{79kgm^2/s}{37kgm^2} = 2.1rad/s$$

Problem 3

A bug of mass m_b = 0.0020kg sits on the rim of a record that rotates with an initial angular velocity of ω = 2π rad/s. The radius of the record is 0.20m and the mass is m_r = 0.10kg. If the bug starts to move and its distance from the center of the record is given by $r = (0.20m)\cos(2t) - (4.0m/s)t$ and it rotates around the center with the same angular velocity as the record, find ω(t).

Solution 3

The moment of inertia of the bug/record system is given by

$$I = I_{bug} + I_{record}$$

The record is treated as a disk with moment of inertia

$$I_{record} = \frac{1}{2} m_r R^2$$

The bug is treated as a point particle

$$I_{bug} = r^2 m = ((0.20m)\cos(2t) - (4.0m/s)t)^2 m_b$$

$$I = ((0.20m)\cos(2t) - (4.0m/s)t)^2 m_b + \frac{1}{2} m_r R^2$$

Angular momentum is conserved

The initial angular momentum is given by (t=0)

$$L = I\omega$$

$$L = ((0.20m)^2 m_b + \frac{1}{2} m_r R^2)\omega = ((0.20m)^2 (0.0020kg) + \frac{1}{2}(0.10kg)(0.20m)^2)(2\pi rad/s) = 0.013kgm^2/s$$

The angular momentum at any time after t=0 has the value 0.013

$$L = I\omega$$

$$\omega(t) = \frac{L}{I(t)}$$

$$\omega(t) = \frac{0.013kgm^2/s}{(0.04m)\cos^2(2t) + (16t^2) - 1.6t^2\cos(2t)}$$

Problem 4

A sphere of radius R=2 sits centered on the origin of a coordinate system. The forces $\vec{F}_1 = 3\hat{i} + 2\hat{j} - 1\hat{k}$, $\vec{F}_2 = 3\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{F}_3 = -1\hat{i} + 0.5\hat{j} - 1\hat{k}$, $\vec{F}_4 = 6\hat{i} + 1\hat{k}$ and $\vec{F}_5 = -3\hat{i}$ are applied to the sphere at positions $\vec{r}_1 = 2\hat{i}$, $\vec{r}_2 = 2\hat{j}$, $\vec{r}_3 = 2\hat{k}$ $\vec{r}_4 = -2\hat{i}$ and $\vec{r}_5 = 0.5\hat{i} + 0.7\hat{k}$. What is the torque about the origin?

Solution 4

Torque

$$\vec{\tau} = \vec{r} \times \vec{F} = (r_y F_z - r_z F_y)\hat{i} + (r_z F_x - r_x F_z)\hat{j} + (r_x F_y - r_y F_x)\hat{k}$$

$$\vec{\tau}_1 = (-(2)(-1))\hat{j} + ((2)(2))\hat{k} = 2\hat{i} + 4\hat{k}$$

$$\vec{\tau}_2 = ((2)(4))\hat{i} + (-(2)(3))\hat{k} = 8\hat{i} - 6\hat{k}$$

$$\vec{\tau}_3 = (-(2)(0.5))\hat{i} + ((2)(-1))\hat{j} = -1\hat{i} - 2\hat{j}$$

$$\vec{\tau}_4 = (-(-2)(1))\hat{k} = 2\hat{k}$$

$$\vec{\tau}_5 = ((0.7)(-3))\hat{j} = -2.1\hat{j}$$

The total torque is just the vector sum of all the torques

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \vec{\tau}_4 + \vec{\tau}_5 = 9\hat{i} - 4.1\hat{j}$$

Problem 5

A disk (M= 5kg, R = 0.5m) rolls along the ground subject to a horizontal force $F = 60N$ applied to its center. What is the acceleration of its center of mass? What is its rotational acceleration?

Solution 5

To find the center of mass motion use Newton's second law for linear motion.

$$\sum F_{ext} = Ma$$

$$\frac{F - f}{M} = a$$

To find is rotational motion use

$$\sum \tau = I\alpha$$

$$fR = I\alpha$$

The moment of inertia for a disk is

$$I = \frac{1}{2}MR^2$$

The rotational equation becomes

$$\frac{2f}{MR} = \alpha$$

The rotational acceleration can be removed with a = Rα

$$\frac{2f}{M} = a$$

The two equations are used to find the center of mass acceleration a

$$\frac{2F}{3M} = a = 8m/s^2$$

The rotational accelleration

$$\frac{a}{R} = \alpha = 16rad/s^2$$