Problem 1
A spring has a spring constant, k = 2.0N/m, and is attached to a mass, m = 1.3kg. If the mass is released with v_0 = 1.5m/s at t = 0s at x = 1.2m (x = 0 is the equilibrium position) what is the equation of motion for the mass?
Solution 1
The mass undergoes harmonic motion. The general equation for harmonic motion is

$$x(t) = A\sin(\omega t + \phi)$$

The angular frequency is given by the equation

$$\omega = \sqrt{k/m} = \sqrt{(2.0N/m)/(1.3kg)} = 1.2rad/s$$

for a spring.
The constants A and ϕ must be determined by the initial conditions. It is known that when t = 0, x = 1.2m

$$1.2m = A\sin(\phi)$$

It is known that the velocity of the mass is 1.5m/s at t = 0. Velocity is the derivative of the position.

$$v(t) = \frac{dx}{dt} = \frac{A}{\omega}\cos(\omega t + \phi)$$

$$1.5m/s = \frac{A}{\omega}\cos(\phi)$$

Dividing the position equation by the velocity equation gives ϕ

$$\frac{1.2m}{1.5m/s} = \frac{A\sin\phi}{(A/\omega)\cos\phi} = \omega\tan\phi$$

$$\phi = \arctan\frac{0.8}{1.2} = 34^0$$

Plug the angle into the position equation to find A

$$1.2m = A\sin(34^o)$$

$$A = \frac{1.2m}{\sin 34^o} = 2.1m$$

A and ϕ have been determined so the equation of motion for the mass is

$$x(t) = A\sin(\omega t + \phi)$$

$$x(t) = (2.1m)\sin(1.2t + 34^o)$$

Problem 2
Given an oscillator with $m = 0.02kg$ and k = 13N/m, what b value will make the amplitude decrease from A to $A/e^{0.92}$ in 1.2s? What b will produce critical damping?
Solution 2
The equation for a damped oscillator is

$$x(t) = Ae^{-(b/2m)t}\cos(\omega' t + \phi)$$

From this equation it can be seen that the amplitude is given in time by

$$Ae^{-(b/2m)t}$$

Set the amplitude at the desired time to the value and solve for b,

$$Ae^{-(b/2m)1.2s} = \frac{A}{e^{0.92}}$$

$$e^{-(b/2m)1.2s} = e^{-0.92}$$

$$-(\frac{b}{2m})(1.2s) = -0.92$$

$$b = \frac{(0.92)(2)(0.02kg)}{1.2s} = 0.031kg/s$$

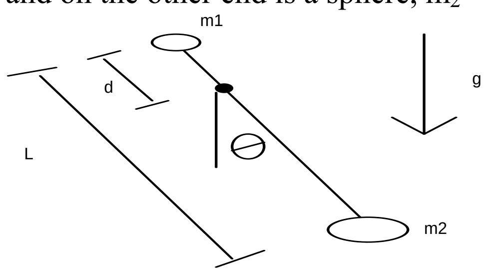
Critical damping is when

$$\frac{k}{m} - \frac{b^2}{4m^2} = 0$$

$$b = 2\sqrt{km} = 2\sqrt{(13N/m)(0.02kg)} = 0.51kg/s$$

Problem 3

A physical pendulum is composed of a rod, L = 0.4m m = 1kg, suspended at a distance, d = 0.1m, from one end and attached to the end nearer to the pivot is a sphere, m_1 = 2kg, and on the other end is a sphere, m_2 = 5kg.



Show that the system undergoes harmonic motion for small oscillations from equibrium. What is the frequency of oscillation.

Solution 3

Use the torque equation

$$\sum \tau = I\alpha$$

The only torque is from gravity, which acts on the center of mass.
The center of mass measured from the pivot point is given by

$$x_{cm} = \frac{m_{rod}x_{cm,rod} + m_1x_1 + m_2x_2}{m_{rod} + m_1 + m_2} = \frac{m_{rod}(L/2-d) + m_1(-d) + m_2(L-d)}{m_{rod} + m_1 + m_2} = 0.23m$$

The torque is given by the component of gravity perpendicular to the rotation arm time the distance from the rotational axis.

$$\tau = (x_{cm})(-Mg\sin\theta) \text{ where M is the total mass}$$

The moment of inertia.
The spheres are treated as point particles

$$I_1 = m_1d^2 = 0.02kgm^2 \qquad I_2 = m_2(L-d)^2 = 0.45kgm^2$$

The moment of inertia for a rod with the axis through one end is $\frac{1}{12}m_{rod}L^2$ from the parallel axis theorem find the moment of inertia about the axis at d.

$$I_{rod} = I_{cm} + m_{rod}(L/2-d)^2 = (1/12)m_{rod}L^2 + m_{rod}(L/2-d)^2 = 0.023kgm^2$$

The moment of inertia for the system is given by

$$I = I_{rod} + I_1 + I_2 = 0.49kgm^2$$

The torque equation is

$$(x_{cm})(-Mg\sin\theta) = I\frac{d^2\theta}{dt^2}$$

For small θ, sinθ = θ.

$$-\frac{x_{cm}Mg}{I}\theta = \frac{d^2\theta}{dt^2}$$

Which is just the harmonic oscillator equation with

$$\omega = \sqrt{\frac{x_{cm}Mg}{I}} = \sqrt{\frac{(0.23m)(1kg + 5kg + 2kg)(9.8m/s^2)}{(0.49kgm^2)}} = 6.1rad/s$$

Problem 4

An angular pendulum made of a disk attached to a torsion rod has a period of T = 0.4s, a mass, m = 0.05kg and a radius, R = 10cm. What is the torsion constant for the rod?

Solution 4

The frequency of an angular pendulum is given by the expression

$$f = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}}$$

Where kappa is the torsion constant

The period is the inverse of the frequency

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{I}{\kappa}}$$

The moment of inertia for a disk is

$$I = \frac{1}{2}mR^2$$

Combing the two expressions

$$T = \pi R\sqrt{\frac{2m}{\kappa}}$$

$$\kappa = \frac{2m\pi^2R^2}{T^2} = \frac{2\pi^2(0.05kg)(0.10m)^2}{(0.4s)^2} = 0.062Nm/rad$$

Problem 5

A mass, m = 23.4g, is attached to a spring, k = 41.2N/m. The system oscillates with an amplitude, A = 8.23cm. What is the total energy of the system? What is the speed of the mass when the displacement is 1.3cm? What is the kinetic and potential energy when the displacement is 4.21cm?

Solution 5

The energy is constant. Any position we choose to evaluate the energy will give the correct value. The easiest place to find the energy is at the max displacement, the kinetic is zero here.

$$E = K + U$$

At the max displacement from equilibrium the energies are

$$E = 0 + \frac{1}{2}kA^2 = \frac{1}{2}(41.2N/m)(0.0823m)^2 = 0.140J$$

From conservation of energy the speed can be determined at any point.

$$E = K + U$$

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$0.140J = \frac{1}{2}(0.0234kg)v^2 + \frac{1}{2}(41.2N/m)(0.013m)^2$$

$$v = 3.41m/s$$

The potential only depends on position, it can be solved for directly

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(41.2N/m)(0.0421m)^2 = 0.0365J$$

Kinetic is found by subtracting the potential from the total energy

$$K = E - U = 0.0140J - 0.0365J = 0.0225J$$