PHY502 Problem Set #4

(1) Lipshutz Problem 6.43

(2) Lipshutz Problem 6.45

(3) Consider a vector space V defined over the field of complex numbers, with $\dim(V) = 2$ and basis $S = \{|v_1\rangle, |v_2\rangle\}$, and a set of linear operators $\{T_0, T_1, T_2, T_3\}$ which act on a vector $|v\rangle = \alpha|v_1\rangle + \beta|v_2\rangle \in V$ as follows:

$$\begin{aligned} T_0(|v\rangle) &= |v\rangle, \\ T_1(|v\rangle) &= \beta|v_1\rangle + \alpha|v_2\rangle, \\ T_2(|v\rangle) &= -i\beta|v_1\rangle + i\alpha|v_2\rangle, \\ T_3(|v\rangle) &= \alpha|v_1\rangle - \beta|v_2\rangle. \end{aligned} \tag{1}$$

Define $T(V)$ to be the set of all linear combinations of $\{T_0, T_1, T_2, T_3\}$:

$$T(V) = \{T : V \to V p; T = \sum_{i=0}^{3} \lambda_i T_i;\ \lambda_i \in \mathbb{C}\}. \tag{2}$$

(a) Show that $\{T_0, T_1, T_2, T_3\}$ forms a basis for $T(V)$.

(b) Find the matrix representation of the set $\{T_0, T_1, T_2, T_3\}$ relative to the basis S.

(c) Find the matrix representation of the set $\{T_0, T_1, T_2, T_3\}$ relative to the basis $S_2 = \{|u_1\rangle, |u_2\rangle\}$, where

$$\begin{aligned} |u_1\rangle &= |v_1\rangle + |v_2\rangle \\ |u_2\rangle &= |v_1\rangle - |v_2\rangle \end{aligned} \tag{3}$$

(d) Show that $T(V)$ is isomorphic the algebra of all linear operators on V, $A(V) = \{A : V \to V\}$.

(e) Show that the matrix representation of the linear operator

$$D_0 : V \to V;\ D_0(\theta) = \exp\left(-\frac{iT_0\theta}{2}\right) \tag{4}$$

relative to the basis S is given by:

$$D_0(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right) & 0 \\ 0 & \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right) \end{pmatrix}. \tag{5}$$

Derive similar expressions for the matrix representations of

$$D_i(\theta) = \exp\left(-\frac{iT_i\theta}{2}\right)\ \ i = 1, 2, 3. \tag{6}$$

(You may use series expansions for functions without proof.)