Topic 6: Cellular Automaton Methods – Continued

The Lattice Boltzmann Equation in the Bhatnagar-Gross-Krook approximation is

$$n_i(\mathbf{r}+\mathbf{c}_i, t+1) = n_i(\mathbf{r}, t) - \frac{n_i - n_i^{\text{eq}}}{\tau} \, .$$

Here τ is a single relaxation time, and n_i^{eq} is the equilibrium configuration, which is given by

$$n_i^{\text{eq}} = \rho w_i \left(1 + \frac{\mathbf{u}\cdot\mathbf{c}_i}{c_s^2} + \frac{(\mathbf{u}\cdot\mathbf{c}_i)^2 - c_s^2\mathbf{u}^2}{2c_s^4} \right) \, ,$$

where c_s is the speed of sound given by

$$\sum_i w_i (\mathbf{c}_i)_a (\mathbf{c}_i)_b = \delta_{ab} c_s^2 \, ,$$

and w_i are a set of directional weights normalized to unity. The discrete lattice velocities and weights are constrained by conservation of mass, momentum, and angular momentum:

$$\sum_i n_i^{\text{eq}} = \rho \, ,$$

where ρ is the fluid density,

$$\sum_i n_i^{\text{eq}} \mathbf{c}_i = \rho \mathbf{u} \, ,$$

where $\mathbf{u}$ is the fluid velocity, and

$$\sum_i n_i^{\mathrm{eq}} (\mathbf{c}_i)_a (\mathbf{c}_i)_b = \rho \left[(\mathbf{u}_i)_a (\mathbf{u}_i)_b + c_s^2 \delta_{ab} \right] \ .$$

The conservation laws imply the following conditions:

$$\sum_i w_i = 1 \ ,$$

$$\sum_i w_i \mathbf{c}_i = 0 \ ,$$

$$\sum_i w_i (\mathbf{c}_i)_a (\mathbf{c}_i)_b = P \delta_{ab} \ ,$$

where P is the fluid pressure.

It can be shown that with these restriction the Navier-Stokes equations are obeyed with the fluid pressure given by

$$P = \rho c_s^2 \ ,$$

and the kinematic viscosity given by

$$\nu = c_s^2 \left(\frac{1}{\omega} - \frac{\delta t}{2} \right) \ ,$$

where $\omega = 1/\tau$ is the relaxation frequency, and δt is the cellular automaton time step which we have choosen to be $\delta t = 1$.

Dr. Succi's program lbe.f

This program simulates 2-D flow in a rectangular region.

The discrete velocities are chosen according to the D2Q9 model is a 2-D lattice (D2) with 9 discrete velocities: 0, N, S, E, W, NE, NW, SE, SW. In the program, `npop = 9` is this number of velocities.

This velocites have

$$c_i = \begin{cases} 0 & \text{for } c = 0 \\ 1 & \text{for N, S, E, W} \\ \sqrt{2} & \text{for NE, NW, SE, SW} \end{cases} .$$

and weights

$$w_i = \begin{cases} \frac{4}{9} & \text{for } c = 0 \\ \frac{1}{9} & \text{for N, S, E, W} \\ \frac{1}{36} & \text{for NE, NW, SE, SW} \end{cases} .$$

The speed of sound is given by

$$2c_s^2 = 1 \times 0^2 \times \frac{4}{9} + 4 \times 1^2 \times \frac{1}{9} + 4 \times (\sqrt{2})^2 \times \frac{1}{36} = \frac{2}{3} .$$