Topic 5: Computational Fluid Dynamics – Particle Suspension

We would like to simulate the steady flow of an incompressible viscous fluid in 2-D. Since the flow is *steady*

$$\frac{\partial \mathbf{v}}{\partial t} = 0 \; .$$

Since the flow is *incompressible*

$$\rho = \text{constant.}$$

The continuity equation

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \; ,$$

becomes

$$\nabla \cdot \mathbf{v} = 0 \; .$$

The Navier-Stokes equations for incompressible viscous steady flow are

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{v} \; ,$$

where ν is the kinematic viscosity. By taking the curl of this vector equation and using

$$\nabla \times \nabla P = 0 \; ,$$

and

$$\nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v}] = \nabla \times \left[\frac{1}{2}\nabla\left(\mathbf{v}^2\right) - \mathbf{v} \times (\nabla \times \mathbf{v})\right] = -\nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})] \; ,$$

the Navier-Stokes equations become

$$\nu\nabla^2(\nabla\times\mathbf{v}) = -\nabla\times[\mathbf{v}\times(\nabla\times\mathbf{v})] \ .$$

Restriction to 2-D: Stream Function and Vorticity

The fluid equations in 2-D can be obtained from the 3-D equations by assuming that

$$v_z = 0 \ , \qquad \frac{\partial\mathbf{v}}{\partial z} = 0 \ .$$

Thus the continuity equation becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \ ,$$

and, using

$$(\nabla\times\mathbf{v})_x = (\nabla\times\mathbf{v})_y = 0 \ , \qquad (\nabla\times\mathbf{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \ ,$$

we see that there is only one non-trivial Navier Stokes equation, which corresponds to the z component

$$\nu\nabla^2(\nabla\times\mathbf{v})_z = -\{\nabla\times[\mathbf{v}\times(\nabla\times\mathbf{v})]\}_z \ .$$

In 3-D, the *vorticity vector field* is defined to be the curl of the velocity $\nabla\times\mathbf{v}$. When restricted to 2-D, this vector only has one non-zero component, which is called the vorticity

$$\zeta = -(\nabla\times\mathbf{v})_z = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \ .$$

The non-zero 2-D velocity components can be derived from a *stream function* ψ as follows:

$$v_x = \frac{\partial \psi}{\partial y} \,, \qquad v_y = -\frac{\partial \psi}{\partial x} \,.$$

This definition guarantees that the continuity equation is satisfied

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \,.$$

The vorticity is determined by

$$\zeta = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = \nabla^2 \psi \,,$$

and using

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = [-\zeta v_y, \zeta v_x, 0] = \left[\zeta \frac{\partial \psi}{\partial x}, \zeta \frac{\partial \psi}{\partial y}, 0\right] \,,$$

we see that the non-trivial Navier-Stokes equation component is

$$\nu \nabla^2 \zeta = \frac{\partial}{\partial x}\left(\zeta \frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\zeta \frac{\partial \psi}{\partial x}\right) = \frac{\partial \zeta}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \zeta}{\partial y}\frac{\partial \psi}{\partial x} \,.$$

Boundary Conditions

We condsider fluid flowing in a region with boundary walls and possibly *inlets* and *outlets*. Suitable boundary conditions must be specified to obtain an unique solution to the fluid flow equations. If the flow is steady, it is sufficient to specify the fluid velocity $\mathbf{v}$ at the boundaries of the region.

Boundary conditions at solid walls: It is necessary that the fluid velocity $\mathbf{v} = 0$ relative to boundary walls. If the wall is at rest the fluid velocity is zero. If the wall is moving, the fluid in contact with the wall moves with the same velocity as the wall.

- We must have $v_\perp = 0$ because a fluid element cannot penetrate the wall, and if it separates from the wall a void will be created which will cause the flow to become turbulent.
- The condition $v_\parallel = 0$ is called the *no-slip condition*. An ideal fluid with zero viscosity can slide frictionlessly past an ideal surface. However, it is found experimentally that even the smallest amount of viscosity or friction in most real fluids results in formation of a *boundary layer*: the fluid in contact with the wall is at rest relative to it, and the fluid velocity increases smoothly with distance from the wall to its free-streaming value inside the region.

Boundary conditions at inlets or outlets: The fluid velocity can be specified arbitrarily at inlets and outlets subject only to conservation of mass. If the fluid is incompressible, then the amount of fluid flowing *into* the region per unit time must necessarily equal the amount of fluid flowing *out of* the region per unit time. This can be expressed as a surface integral on the normal velocity component:

$$\int_{\mathcal{S}} \mathbf{v} \cdot \mathbf{n} \, dS = 0 \, .$$

Steady Flow Through a Square Cavity

Consider a square cavity of side L with an inlet of width w_{in} on its left wall and an outlet of width w_{out} on its bottom wall. Conservation of mass implies that

$$\bar{v}_{\text{in}} w_{\text{in}} = \bar{v}_{\text{out}} w_{\text{out}} \ ,$$

where $\bar{v}_{\text{in,out}}$ are the average velocities normal to the inlet and outlet. We could take the in- and out-flow velocities to be constant across the inlet and outlet. However, this would violate the no-slip condition at the outlet edges. A more realistic velocity profile would rise from 0 at the edges to a maximum near the center of the inlet or outlet. One simple possibility is a parabolic profile. Consider the left wall inlet for example, and let

$$\tilde{y} = \frac{y - y_{\text{mid}}}{\frac{1}{2}w} \ ,$$

where y_{mid} is the location of the midpoint of the inlet and w is its width. The velocity profile can be taken to be

$$v_x = \frac{3}{2}\bar{v}\left[1 - \tilde{y}^2\right] \ ,$$

which can be derived from the stream function

$$\psi = \frac{1}{4}\bar{v}w\left[2 + 3\tilde{y} - \tilde{y}^3\right] \ .$$

We have defined ψ so it varies from $\psi = 0$ at the bottom of the inlet to $\psi = \bar{v}w$ at the top.