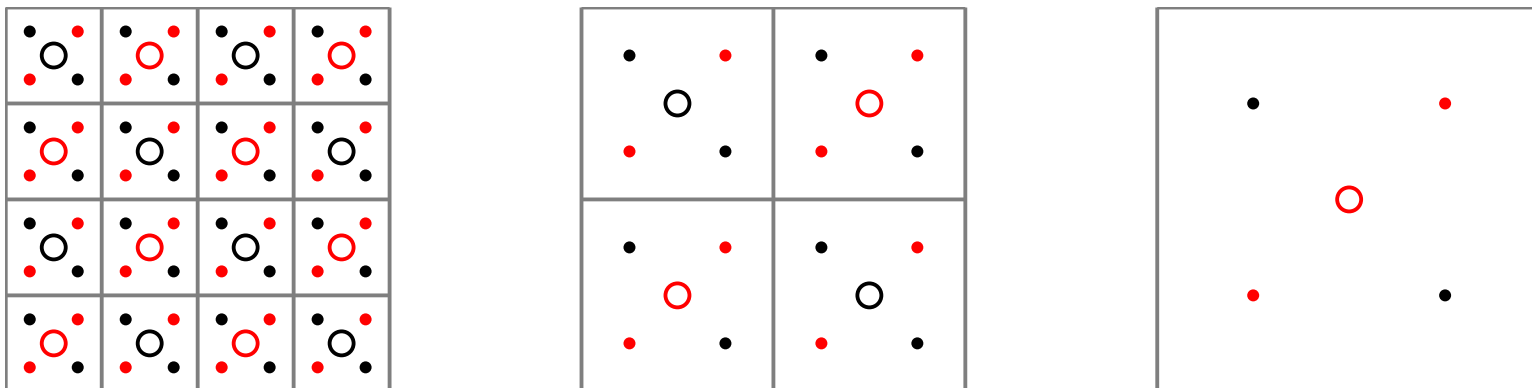


Topic 4: Multigrid Methods (continued)

It is relatively straightforward to adapt the multigrid method to a parallel processing environment. The algorithm uses a sequence of grids instead of the single grid used in the Jacobi, Gauss-Seidel, and SOR algorithms. There is some arbitrariness in the definition of a discretization grid: there are different types of possible grids, and also different possible *coarsening* prescriptions to obtain the sequence of grids. Since the grid points will be allocated to different processors, it is necessary to specify the selected grids and coarsenings precisely.

Cell-centered and Vertex-centered Grids and Coarsenings

In the *cell-centered* prescription, the spatial domain is partitioned into discrete cells. Lattice points are defined at the *center* of each cell as shown in the figure:



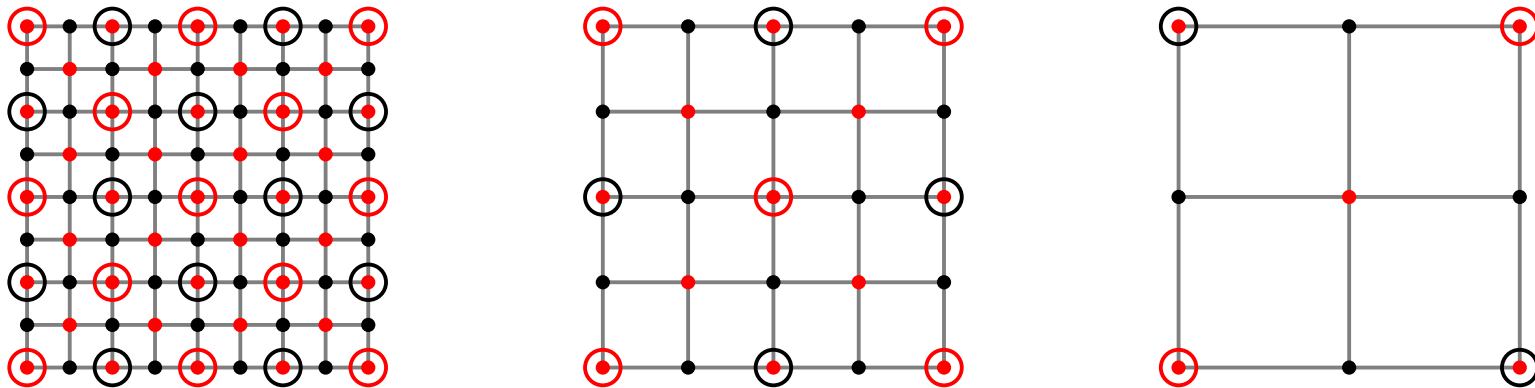
The coarsening operation is defined by doubling the size of a cell in each spatial dimension and placing a coarse lattice point at the center of the doubled cell.

Note that the number of lattice points or cells in each dimension must be a power of 2 if the coarsening operation is to terminate with a single cell. In the figure, the finest lattice has $2^3 = 8$ cells in each dimension, and 3 coarsening operations reduce the number of cells in each dimension

$$2^3 = 8 \rightarrow 2^2 = 4 \rightarrow 2^1 = 2 \rightarrow 2^0 = 1 .$$

Note also that with the cell-centered prescription, the spatial location of lattice sites *changes* with each coarsening: coarse lattice sites are spatially displaced from fine lattice sites.

A *vertex-centered* prescription is defined by partitioning the spatial domain into discrete cells and locating the discrete lattice points at the *vertices* of the cells as shown in the figure:



The coarsening operation is implemented simply by dropping every other lattice site in each spatial dimension.

Note that the number of lattice points in each dimension must be one greater than a power of 2 if the coarsening operation is to reduce the number of cells to a single coarsest cell. In the example in the figure the finest lattice has $2^3 + 1 = 9$ lattice sites in each dimension, and 2 coarsening operations reduce the number of vertices in each dimension

$$2^3 + 1 = 9 \rightarrow 2^2 + 1 = 5 \rightarrow 2^1 + 1 = 3 .$$

The vertex-centered prescription has the property that the spatial locations of the discretization points are not changed by the coarsening operation.

Boundary points

Let's assume that the outermost perimeter points are taken to be the boundary points. The behavior of these boundary points is different in the two prescriptions:

Cell-centered Prescription: The boundary points move in space towards the center of the region at each coarsening. This implies that one has to be careful in defining the “boundary values” of the solution.

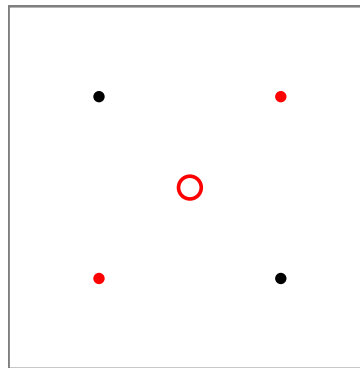
Vertex-centered Prescription: The boundary points do not move when the lattice is coarsened. This make it easier in principle to define the boundary values.

These two different behaviors of the boundary points make the vertex-centered prescription a little more convenient to use in multigrid applications. However, there is no reason why the cell-centered prescription should not work as well.

Restriction and Prolongation Operators

In the multigrid method it is necessary to move functions from a fine grid to the next coarser grid (*Restriction*), and from a coarse grid to the next finer grid (*Prolongation*). Many prescriptions for restricting and prolongating functions have been studied. Let's consider two of the simplest prescriptions appropriate for cell- and vertex-centered coarsening:

Cell-centered Coarsening: In this prescription, a coarse lattice point is naturally associated with 2^d neighboring fine lattice points in d -dimensions.



Suppose that $f(\vec{x})$ is a function on the fine lattice at spatial position \vec{x} , and $F(\vec{X})$ is the corresponding function on the coarse lattice, then this diagram suggests a simple prescription for restriction and prolongation.

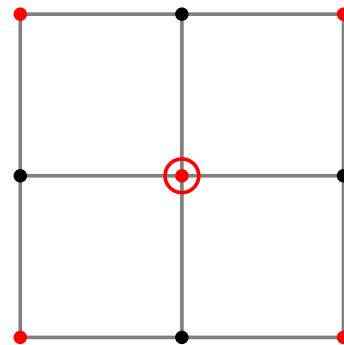
Restriction: Average the function values at the 4 neighboring fine lattice sites \vec{x}_i :

$$F(\vec{X}) = \frac{1}{4} \sum_{i=1}^4 f(\vec{x}_i) .$$

Prolongation: Inject the value of the function at the coarse lattice site to the 4 neighboring fine lattice sites:

$$f(\vec{x}_i) = F(\vec{X}) , \quad i = 1 \dots 4$$

Vertex-centered Coarsening: Consider a coarse lattice point and the 9 neighboring fine lattice points shown in the figure:



In this prescription, a coarse lattice point can naturally associated (in 2-D) with

- the corresponding fine lattice point, or
- the four nearest neighbor fine lattice points, left, right, up, and down, or

- with the four diagonally nearest fine lattice points, etc.

It is a little more complicated here to define transfer operators. The problem is that the fine lattice points are associated with *more than one* coarse lattice point, unlike the cell-centered case:

- The single red fine lattice point in the center coincides with an unique coarse lattice point.
- Each of the 4 black fine lattice points however is equidistant from *two* coarse lattice points.
- Each of the 4 red fine lattice points is equidistant from *four* coarse lattice points.

This sharing of lattice points suggests the following prescriptions:

Prolongation: use *bilinear interpolation* in which the value of F at a coarse grid point is copied to 9 neighboring fine-grid points with the following weights:

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} .$$

This matrix is called the *stencil* for the prolongation.

Restriction: The restriction operator is taken to be the *adjoint* of the prolonga-

tion operator:

$$\begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{bmatrix} .$$