6 Discrete Symmetries: C, P and T

6.1 Charge Conjugation, C

In our "nucleon"-meson theory, we noticed that the amplitude for $N(p_1) + \phi(p_2) \to N(p_1') + \phi(p_2')$ was the same as for $\overline{N}(p_1) + \phi(p_2) \to \overline{N}(p_1') + \phi(p_2')$. The same is true for other processes in this theory which differ only by the exchange of particles for antiparticles, such as $NN \to NN$ and $\overline{NN} \to \overline{NN}$. It is also true to all orders in perturbation theory, not only at $\mathcal{O}(g^2)$, and arises because the theory has an additional symmetry which we have neglected until now, called charge conjugation and denoted C. This is a symmetry transformation which interchanges particles with their antiparticles. Unlike the previous symmetries we have seen, it is not a continuous symmetry parameterized by some continuously varying parameter, and it does not correspond to a conserved current. Instead, it is a *discrete* symmetry.

Given an arbitrary state $|N(\vec{k}_1), \overline{N}(\vec{k}_2), ..., N(\vec{k}_n)\rangle$ composed of nucleons and anti-nucleons we can define a unitary operator U_c which effects this discrete transformation. Clearly,

$$U_c|N(\vec{k}_1), \overline{N}(\vec{k}_2), ..., N(\vec{k}_n)\rangle = |\overline{N}(\vec{k}_1), N(\vec{k}_2), ..., \overline{N}(\vec{k}_n)\rangle. \tag{6.1}$$

We also see that with this definition $U_c^2 = 1$, so $U_c^{-1} = U_c^\dagger = U_c$. We can now see how the fields transform under C. Consider some general state $|\,\psi\rangle$ and its charge conjugate $|\,\overline{\psi}\rangle$. Then $b_k^\dagger|\,\psi\rangle \equiv |N(k), \psi\rangle$, and

$$U_c b_k^\dagger|\,\psi\rangle = U_c|N(k), \psi\rangle = |\overline{N}(k), \overline{\psi}\rangle = c_k^\dagger|\,\overline{\psi}\rangle = c_k^\dagger U_c|\,\psi\rangle. \tag{6.2}$$

Since this is true for arbitrary states $|\,\psi\rangle$, we must have $U_c b_k^\dagger = c_k^\dagger U_c$, or

$$b_k^\dagger \to b_k'^\dagger = U_c b_k^\dagger U_c^\dagger = c_k^\dagger, \quad c_k^\dagger \to c_k'^\dagger = U_c c_k^\dagger U_c^\dagger = b_k^\dagger. \tag{6.3}$$

A similar equation is true for annihilation operators, which is easily seen by taking the complex conjugate of both equations. As expected, the transformation exchanges particle creation operators for anti-particle creation operators, and vice-versa. Expanding the fields in terms of creation and annihilation operators, we immediately see that

$$\psi \to \psi' = U_c \psi U_c^\dagger = \psi^\dagger, \quad \psi^\dagger \to \psi'^\dagger = U_c \psi^\dagger U_c^\dagger = \psi \tag{6.4}$$

It is clearly a symmetry of the theory, $U_c L U_c^\dagger = L$, since ψ and $\psi^\dagger$ always occur together in each term of the Lagrangian. It is straightforward to show that scattering matrix elements are therefore unchanged by charge conjugation. Denoting the initial and final states by $|\,i\rangle$ and $\langle f\,|$ and their charge conjugates by $|\,\bar{\imath}\rangle$ and $\langle \overline{f}\,|$,

$$\begin{aligned} \langle f\,|S|\,i\rangle &= \langle f\,|U_c^\dagger U_c S U_c^\dagger U_c|\,i\rangle \\ &= \langle \overline{f}\,|U_c S U_c^\dagger|\,\bar{\imath}\rangle \\ &= \langle \overline{f}\,|S|\,\bar{\imath}\rangle \end{aligned} \tag{6.5}$$

since $S = T\exp(i\int d^4x\mathcal{H}_I)$, and so $U_c\mathcal{H}_IU_c^\dagger = \mathcal{H}_I \Rightarrow U_cSU_c^\dagger = S$.

While we're at it, there are two other discrete symmetries of $\mathcal{L}_I$ which will be useful in other contexts (it is perhaps worth pointing out here that none of these three symmetries is particularly interesting in this simple theory we are studying. However, they will be much more interesting later on, when we study theories with spin 1/2 and spin 1 fields). These are *parity* (P) and *time reversal* (T). We will look at these in turn.

6.2 Parity, P

A parity transformation corresponds to a reflection of the axes through the origin, $\vec{x} \to -\vec{x}$. Similarly, momenta are reflected, so

$$U_p|\vec{k}\rangle = |-\vec{k}\rangle \tag{6.6}$$

where U_p is the unitary operator effecting the parity transformation. Clearly we will also have

$$U_p \left\{ \begin{array}{c} a_{\vec{k}} \\ a^\dagger_{\vec{k}} \end{array} \right\} U_p^\dagger = \left\{ \begin{array}{c} a_{-\vec{k}} \\ a^\dagger_{-\vec{k}} \end{array} \right\} \tag{6.7}$$

and so under a parity transformation the fields have the transformation

$$\begin{aligned} \phi(\vec{x},t) &\to U_p\phi(\vec{x},t)U_p^\dagger \\ &= U_p\int\frac{d^3k}{\sqrt{2\omega_k}(2\pi)^{3/2}}\left[a_k e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right]U_p^\dagger \\ &= \int\frac{d^3k}{\sqrt{2\omega_k}(2\pi)^{3/2}}\left[a_{-\vec{k}} e^{i\vec{k}\cdot\vec{x}-i\omega_k t} + a^\dagger_{-\vec{k}} e^{-i\vec{k}\cdot\vec{x}+i\omega_k t}\right] \\ &= \int\frac{d^3k}{\sqrt{2\omega_k}(2\pi)^{3/2}}\left[a_k e^{-i\vec{k}\cdot\vec{x}-i\omega_k t} + a_k^\dagger e^{i\vec{k}\cdot\vec{x}+i\omega_k t}\right] \\ &= \phi(-\vec{x},t) \end{aligned} \tag{6.8}$$

where we have changed variables $\vec{k} \to -\vec{k}$ in the integration. Just as before, $U_pLU_p^\dagger = L$, so our theory conserves parity.

Actually, this transformation $\phi(\vec{x},t) \to \phi(-\vec{x},t)$ is not unique. Suppose we had a theory with an additional discrete symmetry $\phi \to -\phi$. This is not true for our nucleon-meson theory, since the interaction term changes sign under this transformation, but it is a symmetry of the Lagrangian $\mathcal{L} = \mathcal{L}_0 - \lambda\phi^4/4!$ which we looked at briefly in the last section. In this case, we could equally well have defined the fields to transform under parity as

$$\phi(\vec{x},t) \to -\phi(-\vec{x},t), \tag{6.9}$$

since that is also a symmetry of $\mathcal{L}$. In fact, to be completely general, if we had a theory of n identical fields $\phi_1 ... \phi_n$, we could define a parity transformation to be of the form

$$\phi_a(\vec{x},t) \to \phi'_a(\vec{x},t) = R_{ab}\phi_b(-\vec{x},t) \tag{6.10}$$

for some $n \times n$ matrix R_{ab}. So long as this transformation is a symmetry of $\mathcal{L}$ it is a perfectly decent definition of parity. The point is, if you have a number of discrete symmetries of a theory there is always some ambiguity in how you define P (or C, or T, for that matter). But this is just a question of terminology. The important thing is to recognize the symmetries of the theory.

In our meson-nucleon theory, $\phi \to -\phi$ is not a symmetry, so the only sensible definition of parity is Eq. (6.8). When ϕ does not change sign under a parity transformation, we call it a *scalar*. In other situations, Eq. (6.8) is not a symmetry of the theory, but Eq. (6.9) is. In this case, we call ϕ a *pseudoscalar*. When there are only spin-0 particles in the theory, theories with pseudoscalars look a little contrived. The simplest example I've seen is

$$\mathcal{L} = \tfrac{1}{2}\sum_{a=1}^{4}\left(\partial^\mu\phi_a\partial_\mu\phi_a - m_a^2\phi_a^2\right) - i\epsilon^{\mu\nu\alpha\beta}\partial_\mu\phi_1\partial_\mu\phi_2\partial_\mu\phi_3\partial_\mu\phi_4 \tag{6.11}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is a completely antisymmetric four-index tensor, and $\epsilon^{0123} = 1$. Under parity, if $\phi_a(\vec{x},t) \to \pm\phi_a(-\vec{x},t)$, then

$$\begin{aligned} \partial_0\phi_a(\vec{x},t) &\to \pm\partial_0\phi_a(-\vec{x},t) \\ \partial_i\phi_a(\vec{x},t) &\to \mp\partial_i\phi_a(-\vec{x},t) \end{aligned} \tag{6.12}$$

where $i = 1, 2, 3$, since parity reverses the sign of $\vec{x}$ but leaves t unchanged. Now, the interaction term in Eq. (6.11) always contains three spatial derivatives and one time derivative because $\epsilon^{\mu\nu\alpha\beta} = 0$ unless all four indices are different. Therefore in order for parity to be a symmetry of this Lagrangian, an odd number of the fields ϕ_a (it doesn't matter which ones) must also change sign under a parity transformation. Thus, three of the fields will be scalars, and one pseudoscalar, or else three must be pseudoscalars and one scalar. It doesn't matter which.

6.3 Time Reversal, T

The last discrete symmetry we will look at is time reversal, T, in which $t \to -t$. A more symmetric transformation is PT in which all four components of x^μ flip sign: $x^\mu \to -x^\mu$. However, time reversal is a little more complicated than P and T because it cannot be represented by a unitary, linear transformation.

We can see why this is the case by going back to particle mechanics and quantizing the Lagrangian

$$L = \frac{1}{2}\dot{q}^2. \tag{6.13}$$

Suppose the unitary operator U_T corresponds to T. Then

$$\begin{aligned} U_T q(t) U_T^\dagger &= q(-t) \\ U_T p(t) U_T^\dagger &= U_T\frac{dq(t)}{dt}U_T^\dagger = -\dot{q}(-t) = -p(-t) \end{aligned} \tag{6.14}$$

and so

$$U_T[q(t), p(t)]U_T^\dagger = U_T i U_T^\dagger = i = -[q(-t), p(-t)] \tag{6.15}$$

and so we cannot consistently apply the canonical commutation relations for all time! Clearly U_T can't be a unitary operator. We need something else.

What we need, in fact, is an operator which is *anti*-linear. Under an antilinear operator Ω,

$$a|\,\psi\rangle \to \Omega\,[a|\,\psi\rangle] = a^*\Omega|\,\psi\rangle. \tag{6.16}$$

That is, numbers are complex conjugated under an antilinear transformation. Since Dirac notation is set up to deal with linear operators, it is somewhat awkward to express antilinear operators in this notation.

The simplest anti-linear operator is just complex conjugation,

$$a|\,\psi\rangle \to \Omega\,[a|\,\psi\rangle] = a^*|\,\psi\rangle \tag{6.17}$$

and in fact this is precisely what we need. First of all, it doesn't screw up the commutation relations because $\Omega i \Omega^{-1} = -i$, so there is an extra minus sign in Eq. (6.15) and there is no contradiction:

$$\Omega_T[aq(t)]\Omega_T^{-1} = a^* q(-t) \tag{6.18}$$

so

$$\Omega_t[q(t), p(t)]\Omega_T^{-1} = i^* = -i = -[q(-t), p(-t)] \tag{6.19}$$

as required. In field theory, complex conjugation corresponds to the operator PT. It has no effect on the creation and annihilation operators,

$$\Omega_{PT} \left\{ \begin{array}{c} a_k \\ a_k^\dagger \end{array} \right\} \Omega_{PT}^{-1} = \left\{ \begin{array}{c} a_k \\ a_k^\dagger \end{array} \right\} \tag{6.20}$$

or on the states

$$\Omega_{PT}|\vec{k}_1, ...\vec{k}_n\rangle = |\vec{k}_1, ...\vec{k}_n\rangle \tag{6.21}$$

(this is to be expected, since time reversal flips the direction of all the momenta, and a parity transformation flips them back). The only thing it acts on is the i in the exponents occurring in the expansion of the fields

$$\begin{aligned}
\phi(\vec{x}, t) &\to \Omega_{PT}\phi(\vec{x}, t)\Omega_{PT}^\dagger \\
&= \Omega_{PT} \int \frac{d^3k}{\sqrt{2\omega_k}(2\pi)^{3/2}} \left[a_k e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}\right] \Omega_{PT}^{-1} \\
&= \int \frac{d^3k}{\sqrt{2\omega_k}(2\pi)^{3/2}} \left[a_k e^{-i\vec{k}\cdot\vec{x} + i\omega_k t} + a_k^\dagger e^{i\vec{k}\cdot\vec{x} - i\omega_k t}\right] \\
&= \phi(-\vec{x}, -t).
\end{aligned} \tag{6.22}$$

Hence this is exactly what is required for a PT transformation.

Now, our meson-nucleon Lagrangian was very dull in that it was invariant under each of these three symmetries separately. For example, the amplitude for

$$N(p_1) + N(p_2) \to N(p_1') + N(p_2') \tag{6.23}$$

was related by C to

$$\overline{N}(p_1) + \overline{N}(p_2) \to \overline{N}(p_1') + \overline{N}(p_2'). \tag{6.24}$$

Under P, this becomes

$$\overline{N}(\omega_1, -\vec{p}_1) + \overline{N}(\omega_2, -\vec{p}_2) \to \overline{N}(\omega_1', -\vec{p}_1') + \overline{N}(\omega_2', -\vec{p}_2'). \tag{6.25}$$

Under T, the incoming and outgoing states are reversed, and the signs of the 3-momenta change sign, so under a T transformation this becomes

$$\overline{N}(p_1') + \overline{N}(p_2') \to \overline{N}(p_1) + \overline{N}(p_2). \tag{6.26}$$

In our theory, the amplitudes for all these processes are identical by the symmetries. In a more general theory, any of C, P or T may be broken. However, it is a *general* property of any local, relativistic field theory that the amplitude must be invariant under the combined action of CPT (this is called the CPT theorem). Hence, while the amplitudes Eq. (6.24) and Eq. (6.25) need not be equal to Eq. (6.23) in some more complicated theory, the amplitudes for Eq. (6.23) and the CPT transformed process Eq. (6.26) will always be the same. Diagrammatically, we can see that this ought to be the case. Consider an arbitrary Feynman diagram with four external nucleon lines, indicated in Fig. (6.1), and

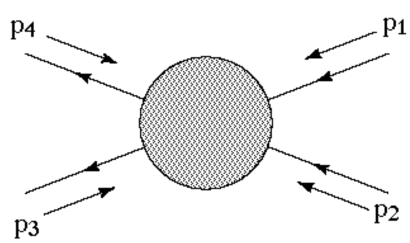

Figure 6.1: A Feynman diagram contributing to the processes Eq. (6.23)–Eq. (6.26). The blob represents an arbitrarily complicated diagram.

define the momenta p_1–p_4 to be directed inward. The blob will be some function $F(p_1, p_2, p_3, p_4)$ of the external momenta. The amplitude for the process (6.23) is then $F(p_1, p_2, -p_1', -p_2')$ while the amplitude for the CPT transformed process, Eq. (6.26) is $F(-p_1, -p_2, p_1', p_2')$ (the diagram doesn't change if we invert it on the page). CPT invariance then simply means that for any diagram,

$$F(p_1, p_2, p_3, p_4) = F(-p_1, -p_2, -p_3, -p_4). \tag{6.27}$$

In a scalar theory, this clearly must be the case. Since F is a Lorentz scalar, it can only depend on the scalar quantities $p_i \cdot p_j$ or (in a more complicated theory) the $\epsilon^{\mu\nu\alpha\beta}$ tensor contracted with the external momenta, $\epsilon^{\mu\nu\alpha\beta} p_\mu p_\nu p_\alpha p_\beta$. All such combinations are invariant under the transformation $p_i \to -p_i$.

The CPT theorem is also true in more general theories with spin, which we will now discuss.