PHY509: FINAL SOLUTION

P1. (a) The kinetic energy $K = \frac{1}{2}m\ell^2(\dot\theta_1^2 + \dot\theta_2^2)$ and the potential energy due to gravity is $V_g = -mg\ell(\cos\theta_1 + \cos\theta_2) \approx -2mg\ell + \frac{1}{2}mg\ell(\theta_1^2 + \theta_2^2)$. With the interaction, the Lagrangian becomes

$$L = \frac{1}{2}m\ell^2(\dot\theta_1^2 + \dot\theta_2^2) - \frac{1}{2}mg\ell(\theta_1^2 + \theta_2^2) + \lambda mg\ell\theta_1\theta_2.$$

(b) The Euler-Lagrange equations of motion become

$$\begin{aligned} \ddot\theta_1 + \frac{g}{\ell}\theta_1 - \frac{g}{\ell}\lambda\theta_2 &= 0 \\ \ddot\theta_2 + \frac{g}{\ell}\theta_2 - \frac{g}{\ell}\lambda\theta_1 &= 0. \end{aligned}$$

Using the form $\theta_i(t) = \theta_{i0}e^{-i\omega t}$, we have an eigenvalue equation

$$\begin{pmatrix} -\omega^2 + g/\ell & -\lambda g/\ell \\ -\lambda g/\ell & -\omega^2 + g/\ell \end{pmatrix} \begin{pmatrix} \theta_{10} \\ \theta_{20} \end{pmatrix} = 0.$$

Solving for roots of the determinant of the above matrix, we have $\omega^2 = g/\ell(1 \pm \lambda)$.
(i) For $\omega = \omega_+ = \sqrt{1+\lambda}\,\omega_0$ ($\omega_0 = \sqrt{g/\ell}$), the eigenvector of the above eigenvalue equation is $(\theta_{10}, \theta_{20}) = \sqrt{2^{-1}}(1, -1)$.
(ii) For $\omega = \omega_- = \sqrt{1-\lambda}\,\omega_0$, the eigenvector is $(\theta_{10}, \theta_{20}) = \sqrt{2^{-1}}(1, 1)$.
(c) The general solution is

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = a_+\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_- t + \phi_-) + a_-\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_+ t + \phi_+).$$

$\phi_i = 0$ is consistent with the initial condition $\dot\theta_i = 0$ at $t = 0$. The condition $\theta_1 = \theta_0, \theta_2 = 0$ at $t = 0$ gives $\theta_0 = \sqrt{2^{-1}}(a_+ + a_-)$ and $0 = \sqrt{2^{-1}}(a_+ - a_-)$. Therefore $a_+ = a_- = \theta_0/\sqrt{2}$. Finally

$$\begin{aligned} \theta_1(t) &= \frac{\theta_0}{2}[\cos(\omega_+ t) + \cos(\omega_- t)] \\ \theta_2(t) &= \frac{\theta_0}{2}[\cos(\omega_+ t) - \cos(\omega_- t)]. \end{aligned}$$

(d) $\theta_2(t) = \theta_0 \sin(\delta\omega t)\sin(\bar\omega t)$ with $\delta\omega = \omega_+ - \omega_-$ and $\bar\omega = (\omega_+ + \omega_-)/2$. The evelope function $\sin(\delta\omega t)$ reaches maximum at

$$t = \frac{\pi}{2\delta\omega} = \frac{\pi}{2(\sqrt{1+\lambda} - \sqrt{1-\lambda})}\sqrt{\frac{\ell}{g}} \approx \frac{\pi}{2\lambda}\sqrt{\frac{\ell}{g}}.$$

P2. (a) $\boldsymbol{\omega}_\alpha = \dot\alpha\hat{\mathbf{z}}_0$, $\boldsymbol{\omega}_\beta = \dot\beta\hat{\mathbf{x}}$, and $\boldsymbol{\omega}_\gamma = \dot\gamma\hat{\mathbf{z}}$. Since $\hat{\mathbf{z}}_0 = \hat{\mathbf{z}}\cos\beta + \hat{\mathbf{y}}\sin\beta$,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\gamma = (\dot\gamma + \dot\alpha\cos\beta)\hat{\mathbf{z}} + \dot\alpha\sin\beta\hat{\mathbf{y}} + \dot\beta\hat{\mathbf{x}}.$$

(b) $K = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}(I_1\omega_x^2 + I_1\omega_y^2 + I_3\omega_z^2) = \frac{1}{2}[I_1(\dot\beta^2 + \dot\alpha^2\sin^2\beta) + I_3(\dot\gamma + \dot\alpha\cos\beta)^2]$. The potential energy is $-Mg\ell\cos\beta$, and finally

$$L = \frac{1}{2}[I_1(\dot\beta^2 + \dot\alpha^2\sin^2\beta) + I_3(\dot\gamma + \dot\alpha\cos\beta)^2] + Mg\ell\cos\beta.$$

(c) α, γ are cyclic coordinates. $p_\gamma = \partial L/\partial\dot\gamma = I_3(\dot\gamma + \dot\alpha\cos\beta)$ and $p_\alpha = \partial L/\partial\dot\alpha = I_1\dot\alpha\sin^2\beta + I_3(\dot\gamma + \dot\alpha\cos\beta)\cos\beta = I_1\dot\alpha\sin^2\beta + p_\gamma\cos\beta$. With the initial conditions $p_\gamma = I_3\omega_0$ and $p_\alpha = I_3\omega_0\cos\beta_0$.

(d) The energy at $t = 0$ is $E = \frac{1}{2}I_3\omega_0^2 - Mg\ell\cos\beta_0$. With the constants p_α, p_γ the energy becomes

$$\frac{1}{2}I_3\omega_0^2 - Mg\ell\cos\beta_0 = \frac{1}{2}I_1\dot\beta^2 + \frac{I_3^2\omega_0^2}{2I_1}\frac{(\cos\beta_0 - \cos\beta)^2}{\sin^2\beta} + \frac{1}{2}I_3\omega_0^2 - Mg\ell\cos\beta.$$

One obvious solution for $\dot\beta = 0$ is $\beta = \beta_0$. The other solution can be obtained by expanding β around β_0 as $\beta = \beta_0 + \Delta\beta$. $\Delta\beta$ is small in the large ω_0 limit. Taylor-expanding the above energy equation with $\dot\beta = 0$, we get

$$0 = \frac{I_3^2\omega_0^2}{2I_1}\frac{(\cos\beta_0 - \cos\beta)^2}{\sin^2\beta} - Mg\ell(\cos\beta - \cos\beta_0) \approx \frac{I_3^2\omega_0^2}{2I_1}\frac{(\sin\beta_0\Delta\beta)^2}{\sin^2\beta_0} - Mg\ell(-\sin\beta_0\Delta\beta).$$

Therefore

$$\Delta\beta = -\frac{2I_1}{I_3^2\omega_0^2}Mg\ell\sin\beta_0.$$

P3. (a) $p_\theta = \partial L/\partial\theta$ and $H = p_\theta\dot\theta - L$. So,

$$H(p_\theta, \theta) = \frac{p_\theta^2}{2m\ell^2} + mg\ell(1 - \cos\theta).$$

(b) With the small angle approximation, $H = p_\theta^2/(2m\ell^2) + \frac{1}{2}mg\ell\theta^2$. Since $P = H$ is just what the Hamilton-Jacobi transformation relates P and H. (It only uses notation α instead of P.) Therefore the generating function $S(\theta, P, t)$ satisfies the equation ($p_\theta = \partial S/\partial\theta$)

$$\frac{1}{2m\ell^2}\left(\frac{dW}{d\theta}\right)^2 + \frac{1}{2}mg\ell\theta^2 = P.$$

$$\begin{aligned}
S(\theta, P, t) &= \int\left[2m\ell^2\left(P - \frac{1}{2}mg\ell\theta^2\right)\right]^{1/2} d\theta \\
Q &= \frac{\partial S}{\partial P} = \sqrt{2m\ell^2}\int\frac{1/2}{\sqrt{P - \frac{1}{2}mg\ell\theta^2}}d\theta = \sqrt{\frac{ell^2}{g}}\int\left(\frac{2P}{mg\ell} - \theta^2\right)^{-1/2} d\theta \\
&= \sqrt{\frac{\ell}{g}}\sin^{-1}\left(\frac{\theta}{\sqrt{2P/mg\ell}}\right).
\end{aligned}$$

It turns out that $\theta = \sqrt{2P/mg\ell}\sin(\omega_0 Q)$ and $p_\theta^2 = 2m\ell^2(P - \frac{1}{2}mg\ell\theta^2) = 2m\ell^2[P - P\sin^2(\omega_0 Q)] = 2m\ell^2 P\cos^2(\omega_0 Q)$ and thus $p_\theta = \ell\sqrt{2mP}\cos(\omega_0 Q)$, which is the same transformation as the example we discussed in class.

(c) $J_\theta = \oint p_\theta d\theta$ and $p_\theta = \partial S/\partial\theta$.

$$J_\theta = \oint p_\theta d\theta = \oint \left[2m\ell^2\left(P - \frac{1}{2}mg\ell\theta^2\right)\right]^{1/2} d\theta = m\ell^2\sqrt{g/\ell}\pi(2P/mg\ell) = 2\pi P\sqrt{\ell/g}.$$

$$\nu = \frac{\partial H}{\partial J_\theta} = \frac{\partial P}{\partial J_\theta} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}.$$

(d) The maximum amplitude θ_{max} happens when $p_\theta = 0$ and $\theta^2_{\text{max}} = 2P/mg\ell$. Expressing P in terms of adiabatic invariants J_θ, we have

$$\theta^2_{\text{max}} = \frac{J_\theta}{\pi m g^{1/2}\ell^{3/2}}.$$

(e)

$$E = \frac{1}{2}mg\ell\theta^2_{\text{max}} = \frac{J_\theta}{2\pi}\sqrt{\frac{g}{\ell}} = \frac{\omega J_\theta}{2\pi}.$$

Therefore,

$$\Delta E = \frac{J_\theta}{4\pi\sqrt{g\ell}}\Delta g.$$