PHY509: Mid-Term Solution

P1. (a) Substituting $F = ku^3$ into the EOM, we get $d^2u/d\phi^2 + u = -(mk/l^2)u$ and therefore $d^2u/d\phi^2 + \alpha^2 u = 0$ with $\alpha = \sqrt{1 + mk/l^2}$. Then we have a general solution,

$$u = A\cos[\alpha(\phi - \phi_0)].$$

(b) The energy conservation has

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2m}u^2 + \frac{k}{2}u^2 = E.$$

Since the equation holds for all ϕ, let's pick $\phi = \phi_0$. Then $\dot{r} = (-1/u^2)(du/d\phi)\dot{\phi} = (-1/u^2)\dot{\phi} \cdot A\alpha\sin[\alpha(\phi_0 - \phi_0)] = 0$ and we have $1/2(l^2/m + k)A^2 = E$ and $A = \sqrt{2E/(l^2/m + k)}$. (c) The scattered particle is at infinity ($u = 0$) at $\cos[\alpha(\phi - \phi_0)] = 0$, i.e, $\phi = \phi_0 \pm \pi/(2\alpha)$. Therefore the angle $2\Delta\phi$ made by the incoming and outgoing asymptotes is π/α. Since the scattering angle θ is

$$\theta = \pi - 2\Delta\phi = \pi\left(1 - \frac{1}{\sqrt{1 + mk/l^2}}\right) = \pi\left(1 - \frac{1}{\sqrt{1 + k/mv_0^2 s^2}}\right).$$

(d) From (c), $s^2 = (k/mv_0^2)[(1 - \theta/\pi)^{-2} - 1]^{-1}$.

$$\sigma(\theta) = \frac{1}{2\sin\theta}\frac{ds^2}{d\theta} = \frac{k}{2mv_0^2\sin\theta}\frac{(2/\pi)/(1 - \theta/\pi)^2}{[(1 - \theta/\pi)^{-2} - 1]^2}.$$

P2. (a) $\mathbf{v} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. The Coriolis force is $-2m\boldsymbol{\omega}\times\mathbf{v} = -2m\omega(\dot{x}\hat{\mathbf{y}} - \dot{y}\hat{\mathbf{x}})$ and the centrifugal force $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = -m\omega^2\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{r}) = -m\omega^2\hat{\mathbf{z}}\times(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}) = m\omega^2(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})$. (b) Drop the centrifugal force and the equation of motion becomes $m\boldsymbol{a} = -mg\hat{\mathbf{r}} - 2m\omega(\dot{x}\hat{\mathbf{y}} - \dot{y}\hat{\mathbf{x}})$. Since the motion is nearly in the $x - z$ plane, we have $\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}}$ and

$$\ddot{x} = -g\sin\theta + 2\omega\dot{y}, \quad \ddot{y} = -2\omega\dot{x}, \text{ and } \ddot{z} = -g\cos\theta.$$

(c) With $\omega = 0$ the motion would have been only in the $x - z$ plane. So we approximate the Coriolis force in the $x - z$ plane as negligible, i.e, $v_x = -gt\sin\theta$ and $v_z = -gt\cos\theta$. Then $\ddot{y} = 2\omega gt\sin\theta$. Solving this equation with the initial condition $y(0) = 0, \dot{y}(0) = 0$ gives $y(t) = \frac{1}{3}gt^3\omega\sin\theta$. The time of flight for the fall is $t = \sqrt{2h/g}$. The object falls eastward ($y > 0$) by the displacement $\frac{1}{3}g(2h/g)^{3/2}\omega\sin\theta$.

P3. (a) The Cartesian coord for the bead is $x = X + R\sin\theta$ and $y = R\cos\theta$. Therefore the kinetic energy of the whole system is

$$\begin{aligned} T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}m[(\dot{X} + R\dot{\theta}\cos\theta)^2 + R^2\dot{\theta}^2\sin\theta] \\ &= \frac{1}{2}(M + m)\dot{X}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}m(R^2\dot{\theta}^2 + 2R\dot{X}\dot{\theta}\cos\theta) \end{aligned}$$

Therefore the Lagrangian is

$$L = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}m(R^2\dot{\theta}^2 + 2R\dot{X}\dot{\theta}\cos\theta) - mgR(1-\cos\theta).$$

(b) With the small angle approximation, we collect terms in L up to the quadratic order of θ, i.e., $\dot{\theta}\cos\theta \approx \dot{\theta}$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Therefore

$$L \approx \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}MR^2\dot{\phi}^2 + \frac{1}{2}m(R^2\dot{\theta}^2 + 2R\dot{X}\dot{\theta}) - \frac{1}{2}mgR\theta^2 + const.$$

The Euler-Lagrange equations for variable X, ϕ, θ are, respectively,

$$\begin{aligned} (M+m)\ddot{X} + mR\ddot{\theta} &= 0 && (1)\\ MR^2\ddot{\phi} &= 0 && (2)\\ R\ddot{\theta} + \ddot{X} + g\theta &= 0 && (3) \end{aligned}$$

Motion of ϕ is irrelevant in the case of no friction because there is no torque exerted on the ring at all. From (1), $\ddot{X} = -mR\ddot{\theta}/(m+M)$ and its substitution into (3) gives

$$\ddot{\theta} - \frac{m}{m+M}\ddot{\theta} + \frac{g}{R}\theta = \frac{M}{m+M}\ddot{\theta} + \frac{g}{R}\theta = 0.$$

Therefore the oscillation frequency is $\sqrt{g(1+m/M)/R}$.

(c) Introducing the Lagrange multiplier $-\lambda(X - R\phi)$ from the constraint $X = R\phi + const.$ results in the modified EOM,

$$\begin{aligned} (M+m)\ddot{X} + mR\ddot{\theta} &= \lambda && (4)\\ MR^2\ddot{\phi} &= -\lambda R && (5)\\ R\ddot{\theta} + \ddot{X} + g\theta &= 0 && (6) \end{aligned}$$

(d) From (4)+(5)/R, we eliminate λ in $(M+m)\ddot{X} + mR\ddot{\theta} + MR\ddot{\phi} = 0$. Using the constraint $\ddot{X} = R\ddot{\phi}$ we get $(2M+m)\ddot{X} + mR\ddot{\theta} = 0$. Following the same steps as (b), we get an equation for θ with the new frequency $\sqrt{g(1+m/2M)/R}$,

$$\frac{2M}{m+2M}\ddot{\theta} + \frac{g}{R}\theta = 0.$$

(e) From (4)-(m+M)(6), $\lambda = mR\ddot{\theta} - (m+M)R\ddot{\theta} - (m+M)g\theta = -MR\ddot{\theta} - (m+M)g\theta = \frac{1}{2}(m+2M)g\theta - (m+M)g\theta = -\frac{1}{2}mg\theta$.