PHY509: SOLUTION HW #9.

P1. (a) With the generating function, $F = F(q,Q)$, we have $p = \partial F(q,Q)/\partial q$ and $P = -\partial F(q,Q)/\partial Q$. Therefore,

$$dp = \frac{\partial^2 F}{\partial q^2}dq + \frac{\partial^2 F}{\partial Q \partial q}dQ = F_{11}dq + F_{21}dQ \qquad (1)$$

$$dP = -\frac{\partial^2 F}{\partial q \partial Q}dq - \frac{\partial^2 F}{\partial Q^2}dQ = -F_{21}dq - F_{22}dQ. \qquad (2)$$

Eliminating dQ from $F_{22} \times (1) + F_{21} \times (2)$ $(F_{21} = F_{12})$, we express dP in terms of dq, dp as

$$F_{22}dp + F_{12}dP = (F_{11}F_{22} - F_{12}^2)dq, \text{ and } dP = \left(\frac{F_{11}F_{22}}{F_{12}} - F_{12}\right)dq - \frac{F_{22}}{F_{12}}dp.$$

Therefore

$$\frac{\partial P}{\partial q} = \frac{F_{11}F_{22}}{F_{12}} - F_{12}, \quad \frac{\partial P}{\partial p} = -\frac{F_{22}}{F_{12}}.$$

From (1), $dQ = -(F_{11}/F_{12})dq + (1/F_{12})dp$ and

$$\frac{\partial Q}{\partial q} = -\frac{F_{11}}{F_{12}}, \quad \frac{\partial Q}{\partial p} = \frac{1}{F_{12}}.$$

Substituting this result into the Poisson bracket we have

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{F_{11}}{F_{12}}\frac{F_{22}}{F_{12}} - \frac{1}{F_{12}}\left(\frac{F_{11}F_{22}}{F_{12}} - F_{12}\right) = 1.$$

(b) $L_x = yp_z - zp_y$, and $L_y = zp_x - xp_z$.

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = y[p_z, z]p_x + p_y[z, p_z]x = -yP_x + p_y x = L_z.$$

Similarly, $[L_i, L_j] = L_k$ with i, j, k in cyclic order.

P2. (a) $[Q,P] = 2c^2(im\omega) = 1$, and thus $C = (2im\omega)^{-1/2} = (2m\omega)^{-1/2}e^{-i\pi/4}$.

(b) Express (p,Q) in terms of (q,P) which S is a function of. First, $p = P/C + im\omega q$. Then $Q = C(p + im\omega q) = P + 2Cim\omega q$. Integrating $p = \partial S/\partial q = P/C + im\omega q$ over q gives $S(q,P) = qP/C + \frac{i}{2}m\omega q^2 + C_1(P)$. Plugging this to the equation for Q, we have $Q = \partial S/\partial P = q/C + dC_1/dP = P + 2Cim\omega q$. Therefore we have $dC_1/dP = P$ and $C_1(P) = \frac{1}{2}P^2$ up to a constant. Finally, $S(q,P) = qP/C + \frac{1}{2}im\omega q^2 + \frac{1}{2}P^2$.

(c) Since $\partial S/\partial t = 0$, $\tilde{H}(Q,P) = H(q,p)$.

$$p = \frac{1}{2C}(P+Q), \quad im\omega q = \frac{Q}{C} - p = \frac{1}{2C}(-P+Q).$$

Therefore

$$\tilde{H}(Q,P) = H(q,p) = \frac{1}{2m}\left[\frac{1}{4C^2}(P+Q)^2 - \frac{1}{4C^2}(-P+Q)^2\right] = \frac{1}{2mC^2}PQ = i\omega PQ.$$

(d) $\dot{P} = -\partial\tilde{H}/\partial Q = -i\omega P$ and $P(t) = P_0 e^{-i\omega t}$ with a complex constant P_0. Similarly, $\dot{Q} = \partial\tilde{H}/\partial P = i\omega Q$ and $Q(t) = Q_0 e^{i\omega t}$. Since the original variable (q,p) should be real at all time, we have, at $t = 0$, $Q_0^* = [C(p + im\omega q)]^* = (2m\omega)^{-1/2} e^{i\pi/4}(p - im\omega q) = i(2m\omega)^{-1/2} e^{-i\pi/4}(p - im\omega q) = iP_0$. Therefore, P_0, Q_0 are chosen to satisfy $Q_0^* = iP_0$ and otherwise arbitrary before any initial conditions are specified.

(e) From (c),

$$\begin{aligned} p &= \frac{1}{2C}(P+Q) = \frac{1}{2C}(P_0 e^{-i\omega t} + Q_0 e^{i\omega t}) \\ q &= \frac{1}{2im\omega C}(-P+Q) = \frac{1}{2im\omega C}(-P_0 e^{-i\omega t} + Q_0 e^{i\omega t}). \end{aligned}$$

Therefore q, p can be expressed in terms of $\sin(\omega t)$ and $\cos(\omega t)$.