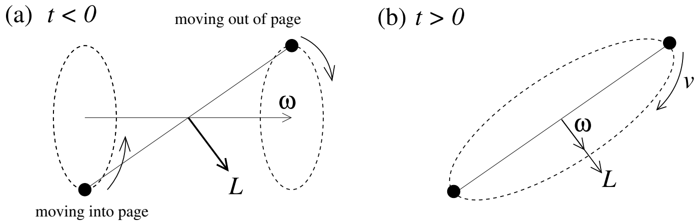
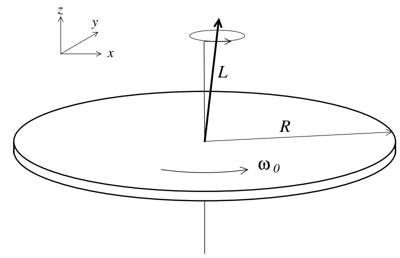
PHY509: SOLUTION HW #8.





P1. (a) The angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$ since the rotation of the whole system along $+\mathbf{x}$ creates the same motion. From the right-hand rule, $\mathbf{L}_i = m_i\mathbf{r}_i \times \mathbf{v}_i$ for each masses points in the direction shown.
(b) From the instant that the constraint force vanishes, the angular momentum becomes a constant vector, i.e., it points in the same direction as (a). And the subsequent motion should be a circular motion of the two masses about the center of mass. The angular velocity and angular momentum are parallel, as shown.

P2. (a) With the mass density $\rho = M/\pi R^2$, $I_z = \int \rho(x^2+y^2)da = \int \rho r^2 \cdot 2\pi r dr = 2\pi\rho R^4/4 = \frac{1}{2}MR^2$. Since $I_x = \int \rho y^2 da = \frac{1}{2}I_z = \frac{1}{4}MR^2 = I_y$.
(b) $\boldsymbol{\omega} = \omega_0\hat{\mathbf{z}}$. $\mathbf{L} = I_x\omega_x\hat{\mathbf{x}} + I_y\omega_y\hat{\mathbf{y}} + I_z\omega_z\hat{\mathbf{z}} = \frac{1}{2}MR^2\omega_0\hat{\mathbf{z}}$.
(c) $\Delta\mathbf{L} = \int \boldsymbol{\Gamma} dt = (L\hat{\mathbf{z}}) \times (F\Delta t\hat{\mathbf{x}}) = FL\Delta t\hat{\mathbf{y}}$. Therefore $\mathbf{L} = \frac{1}{2}MR^2\omega_0\hat{\mathbf{z}} + FL\Delta t\hat{\mathbf{y}}$. Since $\Delta\mathbf{L} = FL\Delta t\hat{\mathbf{y}} = I_y\omega_y\hat{\mathbf{y}}$, $\boldsymbol{\omega} = \omega_0\hat{\mathbf{z}} + (4FL\Delta t/MR^2)\hat{\mathbf{y}}$.
(d) $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} = \frac{1}{2}(I_z\omega_0^2 + I_y\omega_y^2) = \frac{1}{4}MR^2\omega_0^2 + 2(FL\Delta t)^2/MR^2$.
(e) With $I_z = \frac{1}{2}MR^2$ and $I_x = I_y = \frac{1}{4}MR^2$, $I_i\dot{\omega}_i - (I_j - I_k)\omega_j\omega_k = 0$ becomes

$$\dot{\omega}_x + \omega_y\omega_z = 0, \quad \dot{\omega}_y - \omega_x\omega_z = 0, \quad \dot{\omega}_z = 0.$$

(f) $\omega_z = const. = \omega_0$. Equations for $\omega_{x,y}$ are $\dot{\omega}_x + \omega_0\omega_y = 0$, $\dot{\omega}_y - \omega_0\omega_x = 0$. Representing $\omega_{x,y}$ as $\xi = \omega_x + i\omega_y$, the equation of motion can be summarized as $\dot{\xi} - i\omega_0\xi = 0$ and thus $\xi(t) = \xi_0 e^{i\omega_0 t}$ with $\xi(t)$ rotating counter-clockwise in the complex plane. Therefore $\boldsymbol{\omega}$ rotates about the body-$\mathbf{z}$ axis with frequency ω_0.
(g) The initial movement of the axle is in the same direction as the impulse direction, $+\hat{\mathbf{x}}$. However, the axle is going to precess about the new angular momentum direction $\mathbf{L}$ which is tilted toward the $\mathbf{y}$-direction as shown here.

If the torque had been exerted continuously (not through an instant impulse as in this problem), the precession loop of the axle about the angular momentum $\mathbf{L}$ becomes infinitesimally small and the axle would look like following the $\mathbf{L}$ vector. Since $\mathbf{L}$ gets tilted continuously along $\mathbf{y}$ with a continuous torque, the axle will be also getting tilted toward $\mathbf{y}$ direction.