

**PHY509: SOLUTION HW #7.**

- P1.** (a) Since the beads are equi-spaced in equilibrium the interval between the beads is  $2\pi R/N$ .  
 (b) With the virtual displacement  $R\delta\eta_l$  from the equilibrium causes the potential energy from interaction with neighbors as

$$\begin{aligned} V(\delta\eta_l) &= \frac{1}{2}k(\eta_l + \delta\eta_l - \eta_{l-1} + 2\pi R/N - a)^2 + \frac{1}{2}k(\eta_{l+1} - \eta_l - \delta\eta_l + 2\pi R/N - a)^2 \\ &= V(\delta\eta_l = 0) + k(2\eta_l - \eta_{l-1} - \eta_{l+1})\delta\eta_l + O(\delta\eta_l^2). \end{aligned}$$

Therefore  $Q_l = -\partial V/\partial\eta_l = k(\eta_{l+1} + \eta_{l-1} - 2\eta_l)$ .

- (c) Since the kinetic energy is just  $T = \frac{1}{2}\sum_l \dot{\eta}_l^2$ , the Euler-Lagrange equation becomes

$$m\ddot{\eta}_l + k(2\eta_l - \eta_{l+1} - \eta_{l-1}) = 0, \text{ for } l = 1, \dots, N$$

with the periodic boundary condition,  $\eta_{N+1} \equiv \eta_1$  and  $\eta_0 \equiv \eta_N$

- (d) Use the form  $\eta_l(t) = \eta_0 e^{i(kl - \omega t)}$  with undetermined wave number  $k$  and frequency  $\omega$ . Substituting this into the EOM, we get  $-m\omega^2 + k(2 - e^{ik} - e^{-ik}) = 0$  and  $\omega^2 = (2k/m)(1 - \cos k) = (4k/m)\sin^2(k/2)$ . By imposing the boundary condition  $\eta_0 = \eta_N$  with the above form  $\eta_l(t)$ , we get  $e^{ikN} = 1$ . Therefore we get allowed  $k$  values as  $k = 2\pi m/N$  with  $m = 0, 1, \dots, N-1$ . Now, given  $k$ , we obtain the frequency by  $\omega = \sqrt{\frac{4k}{m}}\sin(k/2)$ . Note that the  $k = \omega = 0$  solution is not an oscillator solution. It corresponds to the uniform rotation motion of all the beads.

- P2.** (a) The Cartesian coordinates of the masses are  $x_1 = l\sin\theta_1, y_1 = -l\cos\theta_1$  and  $x_2 = l\sin\theta_1 + l\sin\theta_2, y_2 = -l\cos\theta_1 - l\cos\theta_2$ . With the small angle approximation, we write  $x_1 = l\theta_1, y_1 = -l + \frac{1}{2}l\theta_1^2$  and  $x_2 = l\theta_1 + l\theta_2, y_2 = -2l - \frac{1}{2}l(\theta_1^2 + \theta_2^2)$ . Now the kinetic energy becomes

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \\ &\approx \frac{1}{2}ml^2 \left[ \dot{\theta}_1^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2 + (\theta_1\dot{\theta}_1)^2 + (\theta_1\dot{\theta}_1 + \theta_2\dot{\theta}_2)^2 \right] \approx \frac{1}{2}ml^2 \left[ \dot{\theta}_1^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2 \right]. \end{aligned}$$

The potential energy up to an arbitrary constant is  $V = mg(y_1 + y_2) = \frac{1}{2}mg(2\theta_1^2 + \theta_2^2) + \text{const.}$  Finally the Lagrangian is  $L = T - V \approx \frac{1}{2}ml^2 \left[ \dot{\theta}_1^2 + (\dot{\theta}_1 + \dot{\theta}_2)^2 \right] - \frac{1}{2}mg(2\theta_1^2 + \theta_2^2)$ .

- (b) The EOMs for  $\theta_i$  are

$$\begin{aligned} 2\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{2g}{l}\theta_1 &= 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l}\theta_2 &= 0 \end{aligned}$$

Using  $\theta_i(t) = \theta_i^0 e^{-i\omega t}$ , we get

$$\begin{aligned} -\omega^2(2\theta_1 + \theta_2) + 2\omega_0^2\theta_1 &= 0 \\ -\omega^2(\theta_1 + \theta_2) + \omega_0^2\theta_2 &= 0, \text{ with } \omega_0^2 = g/l. \end{aligned}$$

From the second eq, we get  $\theta_2 = \theta_1 \omega^2 / (\omega_0^2 - \omega^2)$  and by substituting this to the first eq,  $2(\omega^2 - \omega_0^2) + \omega^4 / (\omega_0^2 - \omega^2) = 0$ . Therefore  $\pm \omega^2 = \sqrt{2}(\omega^2 - \omega_0^2)$  and  $\omega_{\pm}^2 = \omega_0^2 / (1 \mp 1/\sqrt{2})$ .

(c) With  $\omega = \omega_{\pm}$ , the eigenvalue equation becomes

$$\begin{pmatrix} \pm\sqrt{2} & 1 \\ 1 & \pm 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0. \quad (1)$$

The solution is  $(\theta_1, \theta_2)_+ = (1, -\sqrt{2})$  and  $(\theta_1, \theta_2)_- = (1, \sqrt{2})$  (without normalization). So we can write a general solution as

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = A_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \cos(\omega_+ t + \phi_+) + A_- \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \cos(\omega_- t + \phi_-).$$

Since  $\dot{\theta}_1 = \dot{\theta}_2 = 0$  at  $t = 0$ , we have  $\phi_+ = \phi_- = 0$  for the same reason as we discussed in class for a textbook example. From the initial condition of angular amplitudes, we have  $\theta_0 = A_+ + A_-$  and  $\theta_0 = \sqrt{2}(-A_+ + A_-)$ . The solution is  $A_+ = \theta_0(\sqrt{2}-1)/2\sqrt{2}$ ,  $A_- = \theta_0(\sqrt{2}+1)/2\sqrt{2}$ . Finally,

$$\begin{aligned} \theta_1(t) &= \frac{\theta_0}{2\sqrt{2}} [(\sqrt{2}-1) \cos(\omega_+ t) + (\sqrt{2}+1) \cos(\omega_- t)] \\ \theta_2(t) &= \frac{\theta_0}{2} [-(\sqrt{2}-1) \cos(\omega_+ t) + (\sqrt{2}+1) \cos(\omega_- t)]. \end{aligned}$$