PHY509: SOLUTION HW #6.

P1. (a) $\partial L/\partial \boldsymbol{v} = m\boldsymbol{v} + q\mathbf{A}$. Therefore,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = m\boldsymbol{a} + q\frac{d\mathbf{A}}{dt} = m\boldsymbol{a} + q\frac{\partial \mathbf{A}}{\partial t} + q(\boldsymbol{v}\cdot\nabla)\mathbf{A}.$$

From the $\partial L/\partial \boldsymbol{r}$ part, we have for the i-th component,

$$\frac{\partial L}{\partial x_i} = -q\partial_i\phi + q\sum_j v_j\partial_i A_j = -q\partial_i\phi + q\partial_i(\boldsymbol{v}\cdot\mathbf{A}).$$

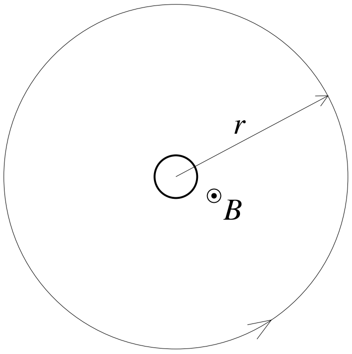
And at the same time,

$$\begin{aligned}\boldsymbol{v}\times\mathbf{B}|_i &= \boldsymbol{v}\times(\nabla\times\mathbf{A})|_i = \epsilon_{ijk}v_j(\nabla\times\mathbf{A})_k = \epsilon_{ijk}v_j\epsilon_{k\alpha\beta}\partial_\alpha A_\beta\\ &= (\delta_{i\alpha}\delta_{j\beta} - \delta_{j\alpha}\delta_{i\beta})v_j\partial_\alpha A_\beta = v_j\partial_i A_j - v_j\partial_j A_i = \nabla(\boldsymbol{v}\cdot\mathbf{A}) - (\boldsymbol{v}\cdot\nabla)\mathbf{A}|_i,\end{aligned}$$

where the permutation symbol ϵ_{ijk} and Kronecker-delta δ_{ij} have been used. Summation is assumed in repeated indices (Einstein convention). Finally,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} = m\boldsymbol{a} + q(\nabla\phi + \partial_t\mathbf{A}) - q\nabla(\boldsymbol{v}\cdot\mathbf{A}) + q(\boldsymbol{v}\cdot\nabla)\mathbf{A} = m\boldsymbol{a} - q\boldsymbol{E} - q\boldsymbol{v}\times\mathbf{B} = 0.$$

(b) Let's turn on $\boldsymbol{A}$ as $\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{\alpha}t$ with a constant vector $\boldsymbol{\alpha}$. Then the electric field $\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial t = -\boldsymbol{\alpha}$ and the magnetic field $\boldsymbol{B} = \nabla\times\boldsymbol{A} = 0$. The Newton's equation has $m\dot{\boldsymbol{v}} = q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) = -q\boldsymbol{\alpha}$. Note that the Newton's equation is $m\dot{\boldsymbol{v}} = \boldsymbol{F}$, NOT with $\dot{\boldsymbol{p}}$ of the canonical momentum $\boldsymbol{p}$ which includes non-kinetic terms. Then $\dot{\boldsymbol{p}} = m\dot{\boldsymbol{v}} + q\partial\boldsymbol{A}/\partial t = m\dot{\boldsymbol{v}} + q\boldsymbol{\alpha} = 0$.



(c) $\boldsymbol{B}$ exists only outside the solenoid. Using that

$$\text{flux } \phi_B = \int_{\text{area}}\boldsymbol{B}\cdot d\boldsymbol{a} = \int(\nabla\times\boldsymbol{A})\cdot d\boldsymbol{a} = \int_{\text{line}}\boldsymbol{A}\cdot d\boldsymbol{s}.$$

Therefore $\phi_B = 2\pi r A_\theta$ or $A_\theta = \phi_B(t)/(2\pi r)$. Turning on the magnetic field at a constant rate, we write $\phi_B(t) = \beta t$. Then $E_\theta = -\partial A_\theta/\partial t = -\beta/(2\pi r)$. Since the motion of the mass is confined to a circular motion, we only consider the angular component of the canonical momentum in the canonical angular momentum, $L_z = r(mv_\theta + qA_\theta)$. The Newton's equation has $m\dot{v}_\theta = qE_\theta + q(\boldsymbol{v}\times\boldsymbol{B})_\theta = qE_\theta = -q\beta/(2\pi r)$.

$$\dot{L_z} = \frac{d}{dt}(rp_\theta) = r(m\dot{v}_\theta + q\dot{A}_\theta) = r\left(-\frac{q\beta}{2\pi r} + q\frac{\beta}{2\pi r}\right) = 0.$$

When the magnetic field changes in time, the induced electro-motive force accelerates the motion of the charge and conserves the canonical angular momentum.

P2. (a) The box gets an instant velocity after the push, but the mass does not get gain a finite change of velocity (in the inerital frame) due to the finite magnitude of force from the spring and the infinitesimally short time interval. Therefore seen with respect to the box, the mass moves with the velocity v_0, $\dot{s}(0) = v_0$.

(b) The coordinate for the mass is $x = X - s$. The kinetic energy of the systems is $T = \frac{1}{2}m(\dot{X} - \dot{s})^2 + \frac{1}{2}M\dot{X}^2$. The Lagrangian is now

$$L = T - V = \frac{1}{2}(m+M)\dot{X}^2 + \frac{1}{2}m\dot{s}^2 - m\dot{X}\dot{s} - \frac{1}{2}ks^2.$$

The EOMs for the variable X and s are, respectively,

$$\begin{aligned} (m+M)\ddot{X} - m\ddot{s} &= 0 \qquad (1) \\ m\ddot{s} - m\ddot{X} + ks &= 0. \qquad (2) \end{aligned}$$

(c) From (1), we have $\ddot{X} = m/(m+M)\ddot{s}$ and after its substitution to (2) $\mu\ddot{s} + ks = 0$, with the reduced mass $\mu = mM/(m+M)$. The oscillation frequency is $\sqrt{k/\mu}$.

(d) The Eq. (1) does not change in the presence of the friction. Any non-conservative force is added to the right hand side of Lagrange-Euler equations, i.e., $m\ddot{s} - m\ddot{X} + ks = -\alpha\dot{s}$.

(e) The terminal velocity is the same as the center of mass velocity of the system since the friction does not exert any external force. Therefore $v_\infty = v_{CM} = Mv_0/(m+M)$. More specifically, by integrating Eq. (1), we get $(m+M)(v_\infty - v_0) = m(\dot{s}_\infty - v_0)$ with $\dot{s}_\infty = 0$. Then we get $v_\infty = Mv_0/(m+M)$.