PHY509: SOLUTION HW #5.

P1. (a) $x = Q + s\cos\alpha$, $z = -s\sin\alpha$ and $X = Q$.

(b) $\delta x = \delta s\cos\alpha$, $\delta z = -\delta s\sin\alpha$ and $\delta X = 0$.

(c) $\delta x = \delta Q$, $\delta z = 0$ and $\delta X = \delta Q$.

(d) δQ does not change the gravitational potential of the system, therefore $Q_Q = -\partial V/\partial Q = 0$. A finite δs changes the potential via δz and $Q_s = -\partial V/\partial s = -\partial(mgz)/\partial s = mg\sin\alpha$.

(e) $T = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) + \frac{1}{2}M\dot{X}^2 = \frac{1}{2}(m+M)\dot{Q}^2 + \frac{1}{2}m(\dot{s}^2 + 2\dot{Q}\dot{s}\cos\alpha)$.
Therefore the equations of motion for Q and s become, respectively,

$$(m+M)\ddot{Q} + m\ddot{s}\cos\alpha = 0 \tag{1}$$

$$m\ddot{s} + m\ddot{Q}\cos\alpha = mg\sin\alpha. \tag{2}$$

(f) The equation (1) is simply the conservation of total linear momentum. From this, we express $\ddot{Q}$ in terms of $\ddot{s}$ as $\ddot{Q} = -(m\cos\alpha/(m+M))\ddot{s}$. Substituting this into Eq. (2) we get

$$\ddot{s} - \frac{m\cos^2\alpha}{m+M}\ddot{s} = g\sin\alpha, \text{ and } \ddot{s} = \frac{(m+M)g\sin\alpha}{M + m\sin^2\alpha}.$$

P2. (a) The length of the pendulum is $l = l_0 - \alpha t$. With the length l and the swing angle θ, $x = l\sin\theta$ and $z = -l\cos\theta$. Therefore the Lagrangian becomes

$$L = T - V = \frac{m}{2}[(\dot{l}\sin\theta + l\dot{\theta}\cos\theta)^2 + (\dot{l}\cos\theta - l\dot{\theta}\sin\theta)^2] + mgl\cos\theta = \frac{m}{2}(\alpha^2 + l^2\dot{\theta}^2) + mgl\cos\theta.$$

(b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(ml^2\dot{\theta}\right) = m(-2\alpha)l\dot{\theta} + ml^2\ddot{\theta}.$$

$$\text{Equation of motion becomes} \quad \ddot{\theta} - \frac{2\alpha}{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0.$$

(c) Since the change in θ is much faster than the change in l, we approximate that l is a constant during a few oscillations. Also assume small oscillation. Then we propose a solution in the form $\theta = \theta_0 e^{-i\sigma t}$, which on substitution to the EOM gives an equation for σ, $\sigma^2 - i(2\alpha/l)\sigma - g/l = 0$. The solution for σ is

$$\sigma = i\frac{\alpha}{l} \pm \sqrt{\frac{g}{l} - \frac{\alpha^2}{l^2}}.$$

$$\text{Finally,} \quad \theta(t) = \theta_0\exp\left[\frac{\alpha t}{l_0 - \alpha t} \pm i\left(\frac{g}{l} - \frac{\alpha^2}{l^2}\right)^{1/2} t\right].$$

The amplitude increases with time, as can be inferred from experience.

Note: Since $\theta(t)$ increases with time, the small angle oscillation we used does not apply after certain time. But, at least, we get some qualitative ideas how it goes.