PHY509: SOLUTION HW #4.

P1. (a) Since the boy travels an angle $\pi/2$ during a time interval t, we have $\omega t = \pi/2$ i.e., $t = \pi/(2\omega)$. The distance the ball travels is $\sqrt{2}R$ and therefore the speed of the ball in the inertial frame is $\sqrt{2}R/t = 2\sqrt{2}R\omega/\pi$. The velocity in the inertial frame is then $\boldsymbol{v}_{\text{inertial}} = (2R\omega/\pi)(\hat{\mathbf{x}}+\hat{\mathbf{y}})$. After transforming this to the body frame at $t = 0$ we have the velocity in the body frame as $\boldsymbol{v}_{\text{body}} = \boldsymbol{v}_{\text{inertial}} - R\omega\hat{\mathbf{x}} = R\omega[(2/\pi - 1)\hat{\mathbf{x}} + 2/\pi\hat{\mathbf{y}}]$. Therefore $v_0 = R\omega\sqrt{(2/\pi - 1)^2 + (2/\pi)^2}$ and $\alpha = \tan^{-1}(-v_x/v_y) = \tan^{-1}(\pi/2 - 1)$.

(b) With $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, $\boldsymbol{\omega} \times \boldsymbol{v} = \omega(\dot{r}\hat{\boldsymbol{\theta}} - r\dot{\theta}\hat{\boldsymbol{r}})$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = -\omega^2 r\hat{\boldsymbol{r}}$. Using these results in the EOM, we get

$$\ddot{r} - r\dot{\theta}^2 = 2\omega r\dot{\theta} + \omega^2 r \text{ and } r\ddot{\theta} + 2\dot{r}\dot{\theta} = -2\omega\dot{r}.$$

(c) The angular part of the above equations becomes $r\ddot{\theta} + 2\dot{r}(\dot{\theta} + \omega) = 0$. By multiplying r to the equation, we get $r^2\ddot{\theta} + 2r\dot{r}(\dot{\theta} + \omega) = (d/dt)[r^2(\dot{\theta} + \omega)] = 0$ and therefore

$$r^2(\dot{\theta} + \omega) = C.$$

At time $t = 0$, $r = R$ and $\dot{\theta} = v_x(t = 0)/R = \omega(2/\pi - 1)$. Therefore $C = 2R^2\omega/\pi$.

(d) The radial part of the EOM is $\ddot{r} - r(\dot{\theta} + \omega)^2 = 0$. Using the result from (c), we get

$$\ddot{r} - \frac{4R^4\omega^2}{\pi^2 r^3} = 0.$$

(e) From (c) $\dot{\theta} = C/r^2 - \omega$. Therefore

$$\Delta\theta = \oint \dot{\theta}dt = \oint \frac{C}{r^2}dt - \omega t.$$

(f) The time interval t can be obtained by $t = \oint dt = \oint(1/\dot{r})dr$. By multipying $\dot{r}$ to eq. of (d), we get

$$0 = \dot{r}\ddot{r} - \frac{C^2}{r^3}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + \frac{C^2}{2r^2}\right) = 0, \text{ and } \dot{r}^2 + \frac{C^2}{r^2} = C'.$$

Using the initial condition $\dot{r} = -v_y = -2R\omega/\pi$ and $C = 2R^2\omega/\pi$, we get $C' = 2(2R\omega/\pi)^2$. And,

$$\dot{r} = \pm\frac{2R\omega}{\pi}\sqrt{2 - \frac{R^2}{r^2}}.$$

The integral limits are $r_{\max} = R$ and $r_{\min} = R/\sqrt{2}$, where $r_{\min}$ is determined from the condition $\dot{r} = 0$. Now the integral can be written as

$$\oint \frac{1}{\dot{r}}dr = 2\int_R^{R/\sqrt{2}} \frac{-\pi}{2R\omega\sqrt{2 - R^2/r^2}}dr = \frac{\pi}{R\omega}\int_{R/\sqrt{2}}^{R} \frac{rdr}{\sqrt{2r^2 - R^2}}.$$

With $y = r^2$,

$$t = \oint \frac{1}{\dot{r}}dr = \frac{\pi}{2R\omega}\int_{R^2/2}^{R^2} \frac{dy}{\sqrt{2y - R^2}} = \frac{\pi}{2R\omega}\sqrt{2y - R^2}\Big|_{R^2/2}^{R^2} = \frac{\pi}{2\omega},$$

in agreement with (a). Now with the $\oint \frac{C}{r^2} dt$,

$$\oint \frac{C}{r^2} dt = \oint \frac{C}{r^2 \dot{r}} dr = 2R \int_{R/\sqrt{2}}^{R} \frac{dr}{r^2 \sqrt{2 - R^2/r^2}} = 2R \int_{1/R}^{\sqrt{2}/R} \frac{du}{\sqrt{2 - R^2 u^2}}.$$

With $u = (\sqrt{2}/R) \sin \phi$,

$$\oint \frac{C}{r^2} dt = 2 \int_{\pi/4}^{\pi/2} d\phi = \frac{\pi}{2}.$$

Therefore $\Delta\theta = \pi/2 - \omega(\pi/2\omega) = 0$.

P2. (a) In the rotating frame, $m\boldsymbol{a} = e(\boldsymbol{E} + c^{-1}\mathbf{v} \times \mathbf{B}) - 2m\boldsymbol{\omega} \times \mathbf{v} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. If we choose $\boldsymbol{\omega} = -(e/2mc)\mathbf{B}$, it cancels the Lorentz force, and $m\boldsymbol{a} = e\boldsymbol{E} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. If the centrifugal term can be much smaller than the Lorentz force, i.e, $(e/c)vB \gg m(eB/2mc)^2 R$ with the velocity v and the radius of the circular orbit as R. With the cancelation of the magnetic force, the condition for the circular orbit is $mv^2/R \approx eE$, and finally

$$B \ll \left(\frac{4mc}{e}\right) \frac{v}{R} = 4\sqrt{\frac{mc^2 E}{eR}}.$$

(b) For a system of particles with the same ratio e/m, the Lorentz force and the Corriolis force cancel the same way.