PHY509: SOLUTION HW #3.

P1. With $f = -ku^3$ plugged into the EOM, we have

$$\frac{d^2u}{d\phi^2} + \left(1 - \frac{mk}{l^2}\right) u = 0.$$

(a) For $mk/l^2 < 1$, the EOM is $u'' + \beta^2 u = 0$ with $\beta^2 = 1 - mk/l^2$ and the solution is sinosoidal, $u(\phi) = A\cos[\beta(\phi - \phi_0)]$.

(b) Since the energy conservation is observed at any angle, let's choose a simple one $\phi = \phi_0$. The potential energy is $V = -\int f dr = -k/(2r^2) = -ku^2/2$ and $u = A$, $u' = 0$ at $\phi = \phi_0$. Then the energy equation becomes

$$E = -\frac{1}{2}kA^2 + \frac{l^2}{2m}A^2, \text{ and } u = \sqrt{\frac{2mE}{l^2\beta}}\cos[\beta(\phi - \phi_0)].$$

Here, ϕ_0 is not determined since it is only about the orientation of the orbit and would not be determined by any conservation laws.

(c) Similarly to (a), the EOM is $u'' - \gamma^2 u = 0$ with $\gamma^2 = mk/l^2 - 1$. The general solution is a linear combination of cosine and sine hyperbolic functions, $u = A\cosh(\gamma\phi) + B\sinh(\gamma\phi)$.

(d) Now pick $\phi = 0$, then $u = A$ and $u' = \gamma B$. Plugging this to the energy equation, we get

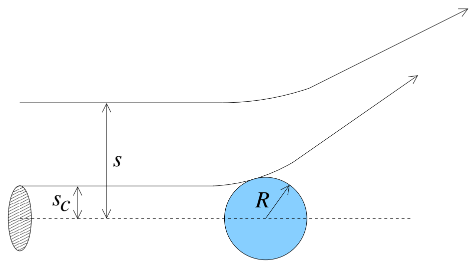
$$E = \frac{l^2}{2m}(\gamma^2 B^2 - mk/l^2 A^2 + A^2) = \frac{l^2}{2m}\gamma^2(B^2 - A^2).$$

(i) For $E > 0$, $|B| > |A|$. Since $|\cosh(\phi_0)| > |\sinh(\phi_0)|$ for any real number ϕ_0, we can write $A = C\cosh(\phi_0)$ and $B = C\sinh(\phi_0)$. With this the solution becomes $u = C[\cosh(\phi)\cosh(\phi_0) + \sinh(\phi)\sinh(\phi_0)] = C\cosh(\phi + \phi_0)$.

(ii) For $E > 0$, $|B| < |A|$. Therefore we write now $A = C\sinh(\phi_0)$ and $B = C\cosh(\phi_0)$ and $u = C[\cosh(\phi)\sinh(\phi_0) + \sinh(\phi)\cosh(\phi_0)] = C\sinh(\phi + \phi_0)$.

(d) $dr/dt = \pm\sqrt{2(E - V - l^2/2mr^2)/m}$ and we are looking at the radius decreasing with time, choose the minus sign. (It was my mistake to omit $\pm$ sign.) Then

$$\begin{aligned} t &= -\int_r^0 \left[\frac{2}{m}\left(E + \frac{k}{2r^2} - \frac{l^2}{2mr^2}\right)\right]^{-1/2} dr = \int_0^r \left[\frac{1}{m}\left(2Er^2 + k - \frac{l^2}{m}\right)\right]^{-1/2} r dr \\ &= \int_0^{r^2} \frac{\sqrt{m}dy}{2\sqrt{2Ey + k - l^2/m}} = \frac{\sqrt{m}}{2E}\sqrt{2Ey + k - \frac{l^2}{m}}\Bigg|_{y=0}^{y=r^2} \\ &= \frac{\sqrt{m}}{2E}\left[\left(k - \frac{l^2}{m} + 2Er^2\right)^{1/2} - \left(k - \frac{l^2}{m}\right)^{1/2}\right] \end{aligned}$$



P2. When the incoming particle passes through the shaded area, it hits the nucleus and a nuclear reaction is expected to happen. Therefore the cross-section for the nuclear reaction is $\sigma_N = \pi s_c^2$. s_c is the critical impact parameter when the particle first hits the nuclear surface. At the contact, the radius is the minimum with $r = R$,

$$\frac{1}{R} = \frac{1}{r_{\text{min}}} = C(-1 + e\cos\phi)_{\text{max}} = C(e-1) \text{ for repulsive force.}$$

Substituting $C = mk/l^2 = k/(2Es^2)$ and $e = \sqrt{1 + (2Es/k)^2}$,

$$\frac{1}{R} = \frac{k}{2Es^2}\left(\sqrt{1 + \left(\frac{2Es}{k}\right)^2} - 1\right).$$

Solving for s^2, we get after some manipulation,

$$s^2 = R^2\left(1 - \frac{k}{RE}\right) \text{ and } \sigma_N = \pi R^2\left(1 - \frac{V_c}{E}\right),$$

with $V_c = k/R$.