PHY509: SOLUTION HW #2.

P1. (a) With $u = 1/r$, $(1/r^2)(dr/d\phi) = -du/d\phi$ and

$$\frac{l^2}{2m}\left(\frac{du}{d\phi}\right)^2 + V(1/u) + \frac{l^2}{2m}u^2 = E.$$

Differentiating w.r.t. ϕ, $(l^2/m)u^{'}u^{''} + (dV/dr)(-1/u^2)u^{'} + (l^2/m)uu^{'} = 0$ and cancelling the commone factor $u^{'}$, we get

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{l^2u^2}\frac{dV}{dr} = -\frac{m}{l^2u^2}f.$$

(b) With $f = -ku^2$, the equation of motion is $d^2u/d\phi^2 + u = mk/l^2$. The special part of the solution is $u_p = mk/l^2$ and the general solution is $u_g = A\cos(\phi - \phi_0)$ and

$$u(\phi) = \frac{1}{r} = \frac{mk}{l^2} + A\cos(\phi - \phi_0),$$

and rewriting $A = -Ce$ and measuring ϕ from ϕ_0, we get $1/r = C(1 - e\cos\phi)$.

(c) Plugging in $u = C(1 - e\cos\phi)$ to the Eq in (a), $LHS = Ce\cos\phi + C(1 - e\cos\phi) = C$. Therefore $f = -(l^2u^2/m)C \propto -1/r^2$.

(d) $f = -dV/dr = -ku^2(1 + 2\alpha ku/mc^2)$. The equation of motion now becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{l^2u^2}\frac{dV}{dr} = \frac{mk}{l^2}(1 + \frac{2\alpha ku}{mc^2}), \text{ and } \frac{d^2u}{d\phi^2} + \left(1 - \frac{2\alpha k^2}{l^2c^2}\right)u = \frac{mk}{l^2}.$$

Adding the particular and general part of the solution, we get

$$\frac{1}{r} = \frac{mk}{\beta l^2} + A\cos(\beta\phi), \text{ where } \beta = \sqrt{1 - \frac{2\alpha k^2}{l^2c^2}}.$$

For the orbital to come back to the same radius after nearly one revolution, the argument fo the cosine function has to change by 2π. i.e., $\beta\Delta\phi = 2\pi$. Therefore, $\Delta\phi = 2\pi/\beta \approx 2\pi + 2\pi\alpha k^2/l^2c^2$. The second contribution $2\pi\alpha k^2/l^2c^2$ corresponds to the advancing precession angle.

P2. (a) For the circular orbit, $l = mv_0R$ and $E = \frac{1}{2}mv^2 - k/R$. With $F = ma$ along the radial direction, $mv^2/R = k/R^2$ and $v = \sqrt{k/mR}$, we get $l = \sqrt{mkR}$ and $E = -k/2R$. After the thrust, the angular momentum does not change because the thrust is also a central force. But the energy change by $\frac{1}{2}mv_0^2$ and $E_{final} = -k/2R + \frac{1}{2}mv_0^2$.

(b) All we need to know is how V_{eff} behaves near $V_{\text{eff}} = E_{\text{min}} = -k/2R$. With $r = R + \delta$,

$$V_{\text{eff}}(r) = -\frac{k}{R+\delta} + \frac{kR}{2(R+\delta)^2} \approx -\frac{k}{R}\left[1 - \frac{\delta}{R} + \frac{\delta^2}{R^2} - \frac{1}{2}\left(1 - \frac{2\delta}{R} + \frac{3\delta^2}{R^2}\right)\right] = -\frac{k}{2R} + \frac{k\delta^2}{2R^3}.$$

The second term matches the energy change at turning points, i.e, $k\delta^2/(2R^3) = \frac{1}{2}mv_0^2$ and $\delta = \pm\sqrt{mv_0^2R^3/k}$. Therefore the closest and farthest radii of the orbit are $R \pm \sqrt{mv_0^2R^3/k}$.

(c) $e = \sqrt{1 + 2El^2/mk^2} = \sqrt{1 + (2R/k)(-k/2R + \frac{1}{2}mv_0^2)} = \sqrt{mR/k}v_0$.

(d) Same as (b) because of the same l and E.