PHY509: SOLUTION HW #10.

P1. (a) Before the change, the kinetic energy K and the potential energy V are related by $K_{\text{before}} = -\frac{1}{2}V_{\text{before}}$ for a circular orbit, as can be also verified by the Virial theorem. Since the change in the mass happens so suddenly that the changes in the resulting velocity and position remain the same. The resulting kinetic energy is the same as K_{before} and the potential changes to half of its initial value due to the change in k (*not* in a). Therefore $E_{\text{after}} = K_{\text{after}} + V_{\text{after}} = K_{\text{before}} + \frac{1}{2}V_{\text{before}} = 0$. The trajectory corresponding to $E = 0$ is a parabola.
(b) The energy in terms of action variables J_r, J_θ is

$$E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta)^2}.$$

In the adiabatic process, J_r, J_θ are invariants. Since k decreases by half, the final energy is a 1/4 of the original energy.
(c) The eccentricity e is expressed in terms of J_r, J_θ as

$$1 - e^2 = -\frac{2El^2}{mk^2} = \frac{4\pi^2 l^2}{(J_r + J_\theta)^2} = \frac{J_\theta^2}{(J_r + J_\theta)^2},$$

with $E = -2\pi^2 mk^2/(J_r + J_\theta)^2$ and $J_\theta = 2\pi l$. Therefore the eccentricity e is also an adiabatic invariant and the circular orbit ($e = 0$) remains a circle.

P2. (a) Since $F = -dV/dx = -f$ for $x > 0$, it creates the same motion as the free-fall in x-direction. Due to the $V = +\infty$, the motion will be bouncing of a mass in a gravitational field. If we set our initial time at the maximum x, the trajectory is $x(t) = x_0 - \frac{1}{2}(f/m)t^2$ with $fx_0 = E$.
(b) Since it takes $\Delta t = \sqrt{2mx_0/f} = \sqrt{2mE}/f$ the period is $2\Delta t = 2\sqrt{2mE}/f$.
(c) For $x > 0$, the Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + fx = \alpha = E.$$

$$J = \oint p dx = \oint \frac{dW}{dx}dx = \sqrt{2m}\oint \sqrt{E - fx}dx = 2\sqrt{2m}\int_0^{E/f} \sqrt{E - fx}dx = \frac{4\sqrt{2mE^3}}{3f}.$$

Therefore the period is $T = 1/\nu = dJ/dE = 2\sqrt{2mE}/f$.
(d) Due to the motion into $x < 0$, the period is doubled. The action integral is also doubled and T becomes twice as large.