PHY509: Hamiltonian Dynamics in a Nutshell

- **Hamilton's principle:** Action I has a stationary value with respect to arbitrary but infinitesimal variations in (q, p),

$$\delta I = \delta \int [p\dot{q} - H(qp)]dt = \int \left[\left(-\frac{\partial H}{\partial q} - \dot{p}\right)\delta q + \left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p\right] dt \Rightarrow \dot{p} = -\frac{\partial H}{\partial q}, \dot{q} = \frac{\partial H}{\partial p}.$$

Notation: Variable without index σ means a collective set, i.e., $q \equiv (q_1, q_2, ..., q_n)$.

- **Canonical transformation** $(qp) \mapsto (QP)$ satisfies, with a generating function F

$$pdq - H(qp)dt = PdQ - \tilde{H}(QP)dt + dF.$$

Choose F to be a function of a pair of independent variables (q, Q), (q, P), (p, Q), (p, P).

$$\text{With } F = F(q, Q, t), \quad pdq - H(qp)dt = PdQ - \tilde{H}(QP)dt + \frac{\partial F}{\partial q}dq + \frac{\partial F}{\partial Q}dQ + \frac{\partial F}{\partial t}dt.$$

By comparing coefficients for dq, dQ, dt,

$$p = \frac{\partial F}{\partial q}, \quad P = -\frac{\partial F}{\partial Q}, \quad \tilde{H}(QP) = H(qp) + \frac{\partial F}{\partial t}.$$

A Legendre-transformed function $S(q, P) \equiv F(q, Q) + PQ$ has

$$pdq - H(qp)dt = -QdP - \tilde{H}(QP)dt + \partial_q S dq + \partial_P S dP + \partial_t S dt$$
$$\Rightarrow \quad p = \frac{\partial S}{\partial q}, \quad Q = \frac{\partial S}{\partial P}, \quad \tilde{H}(QP) = H(qp) + \frac{\partial S}{\partial t}.$$

- Find a generating function $S(q, P, t)$ which makes $\tilde{H}(QP) = H(qp) + \partial S/\partial t = 0$. $\dot{P} = -\partial\tilde{H}/\partial Q = 0$ and replace $P = \alpha$ in S. Since $p = \partial S(q, \alpha, t)/\partial q$, we have the **Hamilton-Jacobi equation**,

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0.$$

With a time-independent Hamiltonian, we separate the time dependence in S with the **Hamilton's characteristic function** $W(q, \alpha)$ as

$$S(q, \alpha, t) = W(q, \alpha) - \alpha_1 t \text{ with } \alpha_1 = E.$$

Then we have the time-independent Hamilton-Jacobi equation

$$H\left(q, \frac{\partial W(q, \alpha)}{\partial q}\right) = \alpha_1.$$

The conjugate variable β to α is also a constant, $\beta = \partial S(q, \alpha, t)/\partial\alpha$.

- Per each q_σ, the **action variable** can be defined as

$$J_\sigma(\alpha) = \oint p_\sigma dq_\sigma = \oint \frac{\partial W_\sigma(q_\sigma, \alpha)}{\partial q_\sigma} dq_\sigma,$$

with the separable Hamilton's characteristic function $W(q, \alpha)$. Since $J_\sigma(\alpha)$'s are also integrals of motion and $\tilde{S}(q, J) = S(q, \alpha)$ satisfies the same Hamilton-Jacobi equation, $\bar{\beta}_\sigma$, the conjugate variable of J_σ, are also constants of motion. Then

$$\bar{\beta}_\sigma = \frac{\partial\tilde{S}}{\partial J_\sigma} = \frac{\partial\tilde{W}}{\partial J_\sigma} - \frac{\partial\alpha_1(J)}{\partial J_\sigma}t \equiv \omega_\sigma(q, J) - \nu_\sigma(J)t,$$

with the **angle variable** $\omega_\sigma(q, J) = \nu_\sigma t + \bar{\beta}_\sigma$ and the **frequency** $\nu_\sigma(J) = \partial\alpha_1(J)/\partial J_\sigma$.

- When the Hamiltonian $H(p, q, \lambda(t))$ changes adiabatically with $\dot{\lambda}/\lambda \ll 1/T$ (T = period), the action variables $J_\sigma = \oint p_\sigma dq_\sigma$ are **adiabatic invariants**.

- Let (α_0, β_0) are canonical variables satisfying the Hamilton-Jacobi equation, i.e, they are constants of motion. With a perturbation ΔH added to the Hamiltonian H, the canonical variables $\alpha_0(pq)$, $\beta_0(pq)$ are no longer constants in time. $(\alpha(pq), \beta(pq))$ satisfies the Hamilton's equation of motion

$$\dot{\alpha} = -\frac{\partial \Delta H}{\partial \beta}, \quad \dot{\beta} = \frac{\partial \Delta H}{\partial \alpha}.$$

The approximate solution is obtained recursively,

$$\begin{aligned}
(\alpha_0, \beta_0) \quad &\rightarrow \quad \dot{\alpha}_1 = -\frac{\partial \Delta H}{\partial \beta}(\alpha_0, \beta_0), \quad \dot{\beta}_1 = \frac{\partial \Delta H}{\partial \alpha}(\alpha_0, \beta_0) \\
(\alpha_1, \beta_1) \quad &\rightarrow \quad \dot{\alpha}_2 = -\frac{\partial \Delta H}{\partial \beta}(\alpha_1, \beta_1), \quad \dot{\beta}_2 = \frac{\partial \Delta H}{\partial \alpha}(\alpha_1, \beta_1) \\
&\quad\ \ \cdots
\end{aligned}$$