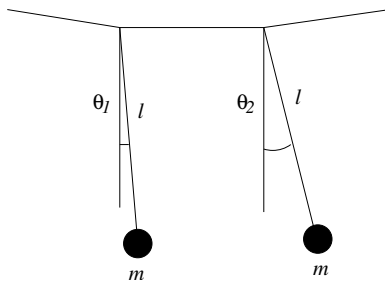


PHY509 Classical Dynamics: Final Exam (12/19/05)



**P1.** Two identical bobs of mass  $m$  are hanging from strings of length  $\ell$ . The two pendulums interact with each other through the another supporting string as shown and we assume that the mechanical energy of the system is conserved. The interaction between the two pendulums is modeled as

$$V_{\text{int}} = -\lambda m g \ell \theta_1 \theta_2,$$

with  $\theta_1, \theta_2$  defined as in the figure.

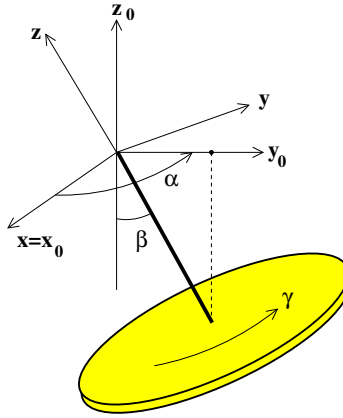
**(30 points)**

(a) Assume small angle oscillation. What is the Lagrangian of the system? **(5 pts)**

(b) What are the oscillation frequency and eigenvectors of eigenmodes? **(10 pts)**

(c) The initial condition is given at  $t = 0$  as  $\dot{\theta}_1 = \dot{\theta}_2 = 0$  and  $\theta_1 = \theta_0$  and  $\theta_2 = 0$ . Solve for  $\theta_1(t)$  and  $\theta_2(t)$ . **(10 pts)**

(d) Find the time needed for the mechanical energy of the pendulum 1 to be transferred to the pendulum 2. Simplify your answer for small  $\lambda$  limit. **(5 pts)**



**P2.** A symmetrical rigid body is suspended through a rod to a point about which it freely rotates. At certain instant ( $t = 0$ ), let us choose our inertial coordinate system such that the projection of the rod (i.e. line of node) coincides with the inertial  $\hat{y}_0$  axis as shown in the figure. The advancing angle of the line of node in the inertial  $(x_0, y_0)$  plane is defined as  $\alpha$  and the polar angle of the rod made from the vertical is  $\beta$  and the spinning angle of the rigid body about the body- $\hat{z}$  is given as  $\gamma$ , as usual. The gravitation acceleration is  $g$  in the negative  $\hat{z}_0$  direction.

**(30 pts)**

(a) Write down the angular velocity vector  $\boldsymbol{\omega}$  with respect to the body axis system,  $(\hat{x}, \hat{y}, \hat{z})$  at time  $t = 0$ . **(10 pts)**

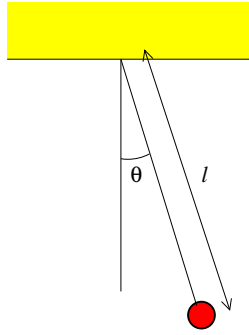
(b) Let the moment of inertial about the body- $z$  axis be  $I_3$  and those about the body- $x, y$  axes be  $I_1$ . What is the kinetic energy in terms of  $(\alpha, \beta, \gamma, \dot{\alpha}, \dot{\beta}, \dot{\gamma})$ ? Write down the Lagrangian of the system. The mass of the rigid body is  $M$  and the length of the (weightless) rod is  $\ell$ . **(5 pts)**

(c) From now on, we specify the initial condition. We hold the rod still such that  $\dot{\alpha} = \dot{\beta} = 0$ ,  $\dot{\gamma} = \omega_0$  and  $\beta = \beta_0$ . At  $t = 0$  we release the system. With this initial condition, evaluate the conjugate momenta  $p_\alpha, p_\gamma$  corresponding to the cyclic variables  $\alpha, \gamma$ . **(5 pts)**

(d) Given the fixed  $p_\alpha, p_\beta$ , one can express the energy in terms of non-cyclic variable  $\beta$  as

$$E = \frac{1}{2}I_1\dot{\beta}^2 + \frac{(p_\alpha - p_\gamma \cos \beta)^2}{2I_1 \sin^2 \beta} + \frac{p_\gamma^2}{2I_3} - mgl \cos \beta.$$

Due to the gravity the polar angle  $\beta$  decreases. What is the maximum change of the angle,  $\Delta\beta$ ? Simplify your answer in the limit of large  $\omega_0$ . **(10 pts)**



**P3.** Consider a simple pendulum with Lagrangian,

$$L(q, \dot{q}) = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta),$$

where  $\theta$  is the oscillating angle from the vertical and  $\ell$  is the length of the string of the pendulum. In this problem we define the potential as measured from the bottom of the swing ( $\theta = 0$ ).

**(40 points)**

(a) Transform the Lagrangian into a Hamiltonian  $H(p_\theta, \theta)$ .

**(5 pts)**

(b) Assume a small angle motion with the maximum angular amplitude  $\theta_0 \ll 1$ . It is then proposed that we find a canonical transformation with the new momentum  $P$  given as

$$P = \frac{p_\theta^2}{2m\ell^2} + \frac{1}{2}mg\ell\theta^2.$$

What is the transformation for the coordinate  $Q$  which is conjugate to  $P$ ?

**(10 pts)**

Hint: Use the generating function  $S(\theta, P, t)$  or Hamilton's characteristic function  $W(\theta, P)$  to obtain  $Q$  as  $Q(\theta, P)$ .

(c) Calculate the action variable  $J_\theta = \oint p_\theta d\theta$  from the Hamilton-Jacobi theory and the frequency  $\nu = \partial H / \partial J_\theta$ .

**(10 pts)**

(d) Imagine that we can turn a knob to change the gravitational acceleration  $g$ . When  $g$  is increased adiabatically, what is the maximum angular amplitude  $\theta_{\max}(g)$  as a function of  $g$ .

**(10 pts)**

(e) Is the energy conserved during the process? If not, what is the energy change  $\Delta E$  due to a small change of the gravitational acceleration  $\Delta g$ ?

**(5 pts)**

Integrals:  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a) + C, \quad \int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi}{4}a^2.$