Perturbation theory in the QCD coupling $\alpha_s(Q)$ can be used only at high energies $Q \gg 1$ GeV. Predictions for low energy properties, such as the spectrum of mesons and baryons, the decays of unstable hadrons, and the properties of the QCD vacuum, require non-perturbative methods. Lattice QCD is the most widely method for numerical computation of low energy properties of QCD.

**Meson and Glueball Spectrum**

See USQCD Physics: structure and interactions of hadrons for details and references.
Kaon and Pion Decay Constants

From Particle Data Group, Reviews Section 17. Lattice Quantum Chromodynamics

![Graph showing $f_K/f_\pi$ values for various collaborations.]

**Figure 17.1:** Results for $f_K/f_\pi$ from simulations with $N_f = 2 + 1$. These are from the HPQCD/UKQCD [29], ALV [58], BMW [60], RBC/UKQCD [61] and MILC [62] collaborations. The resulting average is $1.1931 \pm 0.0053$.

Charged Pions and Kaons by $q\bar{q}$ annihilation to a $W$ boson, see Griffiths, Introduction to Elementary Particles, Section 9.4 Pion Decay.
Bound States of Heavy Quarks

The figure on the left shows a fit to a Wilson loop calculation of the potential $V(R)$ between a quasi-static heavy quark-antiquark pair

$$V(R) = C + \frac{B}{R} + \sigma R + \lambda \left( \frac{1}{R} \right)_{\text{lat}} \left( \frac{1}{R} \right)$$

showing the short distance 3-D Coulomb $1/R$ and long distance 1-D linear $R$ behavior. The figure on the right shows lattice QCD results for the bound state spectrum of $c\bar{c}$ pairs Charmonium.

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The figures from M. Creutz, *Phys. Rev. Lett.* 45, 313-316 (1980) show that QCD at zero temperature has a single phase with confined color. The effective potential changes continuously from linear behavior $V(R) \sim R$ at strong coupling to asymptotically free Coulomb behavior $V(R) \sim 1/R \log(Q)$ at weak coupling.
Summary of Lepage’s Lattice QCD for Novices

Section 4 of G.P. Lepage, Lattice QCD for Novices has a clear discussion of the basic concepts and suggestions for writing a minimal code that should give reasonable results on a personal computer.

The Classical Action and Gauge Invariance

The continuum action for QCD is

$$S = \int d^4 x \frac{1}{2} \sum_{\mu, \nu} \text{Tr} F_{\mu \nu}^2(x)$$

where

$$F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

is the field tensor, a traceless $3 \times 3$ hermitian matrix. The theory is invariant with respect to gauge transformations where

$$F_{\mu \nu} \rightarrow \Omega(x) F_{\mu \nu} \Omega(x)^\dagger$$

and $\Omega(x)$ is an arbitrary $x$-dependent SU(3) matrix.

See Wikipedia Special unitray group for more on SU(3) matrices.

Link Variables and Wilson Loops

The action is discretized by specifying the gauge field $A_\mu(x)$ on the links joining the sites of a hypercubic 4-D lattice. In the classical theory, the “link variable” on the link joining a site at $x$ to
one at $x + a\hat{\mu}$ is determined by the line integral of $A_\mu$ along the link:

$$U_\mu(x) \equiv \mathcal{P} \exp \left( -ig \int_x^{x + a\hat{\mu}} A \cdot dy \right)$$

where the $\mathcal{P}$-operator path-orders the $A_\mu$’s along the integration path. $U_\mu$ instead of $A_\mu$ on the lattice, because it is impossible to formulate a lattice version of QCD directly in terms of $A_\mu$’s that has exact gauge invariance. The $U_\mu$’s, on the other hand, are SU(3) matrices that transform very simply under a gauge transformation:

$$U_\mu(x) \rightarrow \Omega(x) U_\mu(x) \Omega(x + a\hat{\mu})^\dagger.$$ 

This makes it easy to build a discrete theory with exact gauge invariance.

A link variable $U_\mu(x)$ is represented pictorially by a directed line from $x$ to $x + \hat{\mu}$, where this line is the integration path for the line integral in the exponent of $U_\mu(x)$, and the conjugate matrix $U_\mu^\dagger(x)$ the direction of the line integral is flipped and so we represent $U_\mu^\dagger(x)$ by a line going backwards from $x + \hat{\mu}$ to $x$. 

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**Diagram:**

- A directed line is shown from $x$ to $x + \hat{\mu}$ labeled $U_\mu(x)$.
- The conjugate matrix $U_\mu^\dagger(x)$ is shown as a line going backwards from $x + \hat{\mu}$ to $x$. 

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The figure on the right shows a Wilson loop function,

\[ W(C) \equiv \frac{1}{3} \text{Tr} \mathcal{P} e^{-i \oint_C g A \cdot dx} \]

for the closed path \( C \) built of links on the lattice, which is defined as path-ordered product of the \( U_\mu \)'s and \( U_\mu^\dagger \)'s associated with each link.

**Plaquettes, Action, and Path Integral**

The lattice action is a straightforward generalization of the abelian Lattice QED action. A discrete Lagrangian density is associated with each elementary square (plaquette) of the lattice:

\[ P_{\mu \nu} = \frac{1}{3} \text{Re} \ \text{Tr} \ \mathcal{P} \ \exp \left[ -ig \oint_{\text{plaquette}} A \cdot dx \right] = \frac{1}{3} \text{Re} \ \text{Tr} \ \Pi_{x \mu} U_\mu (x), \]

where the product is taken around plaquette using \( U \) or \( U^\dagger \) depending on the direction of the arrow.
The lattice action is the sum over plaquette Lagrangians

\[ S = \sum_{\mu \nu} (1 - P_{\mu \nu}) \]

over all plaquettes on the lattice.

The path integral (partition function) is then given by integrating over the link variables

\[ Z(\beta) = \int \Pi x_\mu dU_\mu(x) \, e^{-\beta S}. \]

**Metropolis Monte Carlo Evaluation of the Path Integral**

A computer code for the Monte Carlo evaluation of QCD path integrals can be designed in close analogy with the abelian QED case.

The Metropolis algorithm for generating random configurations must be adapted to work with SU(3) matrices. In the QED application the photon field was represented by angular variables \( \theta_\mu(n) \) on the links. In QCD the gluon field is specified by link variables \( U_\mu(x) \), which are exponentials of the fundamental gluon field.

The simplest 3-dimensional defining **Representation of SU(3)** can be used to represent the \( M \)'s. To update a link variable, multiply it by a random SU(3) matrix \( M \):

\[ U_\mu \rightarrow M U_\mu. \]
Typically the matrix $M$ is chosen randomly from a set of 50 or 100 random SU(3) matrices that is generated once, at the start of the simulation; the only restrictions on this set are that it be large enough so that products of the various $M$’s cover the entire space of SU(3) matrices, and that the inverse, $M^\dagger$, for each matrix $M$ in the set also be included in the set.

**Generating Random SU(3) Matrices**

The $M$’s can be generated by first creating a set of hermitian matrices $H$ whose matrix elements are random numbers between $-1$ and $1$. These are converted to SU(3) matrices by forming $1 + i\epsilon H$ and unitarizing it. The parameter $\epsilon$ is the Metropolis trial step size, which is adjusted so that roughly half of all trial updates are accepted.

To convert an arbitrary matrix $M = (m_1 m_2 m_3)$ into an SU(3) matrix:

- Normalize the first column $m_1$ to unity.
- Make the second column $m_2$ orthogonal to the (new) first column and normalize it.
- Replace the third column by the cross product of the (new) first two columns.