Chaos in the Double Pendulum

Chaotic Trajectories of ODE Systems

What is the smallest number of coupled ordinary differential equations that exhibits chaotic behavior? This depends on whether or not the equations depend explicitly on time. If the equations depend on time, then the minimum number is two. If they do not, at least three equations are required for chaotic behavior to occur.

Chaos in the kicked rotor, which equivalent to the standard map, is a result of a periodic kicking force that depends on the angular position of the rotor.

Chaos in the nonlinear driven damped pendulum results from the competition between nonlinearity, driving, and damping, with the magnitude of the damping force being the critical parameter.

The phase spaces of the rotor and pendulum are 2-dimensional. In both systems, the equations depend explicitly on time through the kicking or driving force terms.

The Lorenz model is a system of three coupled first order ordinary differential equations which has chaotic trajectories.

The phase space of a Hamiltonian system has an even number of dimensions because each generalized coordinate is paired with its canonically conjugate generalized momentum. If the Hamiltonian does not depend on time, then the energy of the system is constant and provides a first integral of motion. Any time-independent Hamiltonian system with a 2-dimensional phase space is integrable. A time-independent Hamiltonian system must have at least a 4-dimensional phase space for chaotic trajectories to occur.

The Double Pendulum

The double pendulum consists of two simple pendulums that are attached to one another as shown in Fig. 1. The top and center pivots are assumed frictionless, and the coupled objects are free to rotate about them in the vertical plane under the action of gravity.

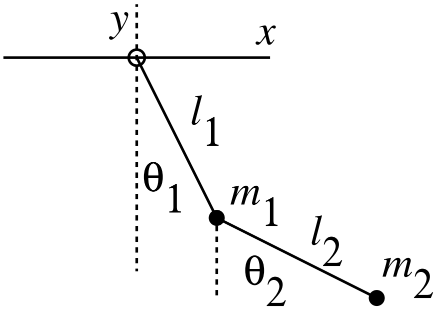

Figure 1: A double pendulum consisting of two simple pendulums.

The double pendulum is the simplest example of a time-independent Hamiltonian system that exhibits chaotic behavior. Its phase space is 4-dimensional.

Chaos can be demonstrated experimentally in the double pendulum[1],[2].

Euler-Lagrange Equations

Suppose that the upper pendulum has a massless rod of length ℓ_1 and a bob of mass m_1. The lower pedulum has a massless rod of length ℓ_2 and bob of mass m_2. The two rods provide constraints on the motion of the two masses in the vertical plane:

$$x_1 = \ell_1 \sin\theta_1 \ , \qquad y_1 = -\ell_1 \cos\theta_1 \ , \tag{1}$$

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \ , \qquad y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 \ . \tag{2}$$

There are therefore only two independent generalized coordinates, which can be taken to be the angles θ_1 and θ_2 that the two rods make with the downward vertical direction.

The Lagrangian for the double pendulum is

$$\begin{aligned} L &= \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) - m_1 g y_1 - m_2 g y_2 & (3)\\ &= \frac{1}{2}(m_1+m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_2\dot{\theta}_2\cos(\theta_1-\theta_2) & (4)\\ &\quad +(m_1+m_2)g\ell_1\cos\theta_1 + m_2 g\ell_2\cos\theta_2 \ . & (5)\end{aligned}$$

The Euler-Lagrange equations for θ_1 and θ_2 are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) = \frac{\partial L}{\partial \theta_i} \ . \tag{6}$$

The θ_1 equation is

$$\ell_1\left[(m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\cos(\theta_1-\theta_2)\ddot{\theta}_2 + m_2\ell_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2 + (m_1+m_2)g\sin\theta_1\right] = 0 \ , \tag{7}$$

and the θ_2 equation is

$$m_2\ell_2\left[\ell_2\ddot{\theta}_2 + \ell_1\cos(\theta_1-\theta_2)\ddot{\theta}_1 - \ell_1\sin(\theta_1-\theta_2)\dot{\theta}_1^2 + g\sin\theta_2\right] = 0 \ . \tag{8}$$

Small Oscillations

When the amplitudes of oscillation are small, the equations of motion can be linearized to find the normal modes:

$$(m_1+m_2)\ell_1\ddot{\theta}_1 + m_2\ell_2\ddot{\theta}_2 + (m_1+m_2)g\theta_1 = 0 \ , \tag{9}$$

$$\ell_2\ddot{\theta}_2 + \ell_1\ddot{\theta}_1 + g\theta_2 = 0 \ . \tag{10}$$

Normal mode solutions have the form:

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \begin{pmatrix}\theta_{10}\\ \theta_{20}\end{pmatrix} e^{i\omega t} \ . \tag{11}$$

The normal mode frequencies are determined by solving the secular equation

$$\begin{vmatrix} m_1(g-\omega^2\ell_1) & -m_2\ell_2\omega^2 \\ -\ell_1\omega^2 & (g-\omega^2\ell_2)\end{vmatrix} = 0 \ . \tag{12}$$

The solutions of the secular equation are:

$$\omega^2 = \left[\frac{g(\ell_1+\ell_2)}{2\ell_1\ell_2}\right]\left[\frac{m_1+m_2 \pm \sqrt{(m_1+m_2)\left[m_2+m_1\left(\frac{\ell_1-\ell_2}{\ell_1+\ell_2}\right)^2\right]}}{m_1}\right] . \tag{13}$$

The normal mode amplitudes are related by

$$\frac{\theta_{20}}{\theta_{10}} = \frac{g-\omega^2\ell_2}{\omega^2\ell_1} . \tag{14}$$

Consider the special case of equal masses and lengths, $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = L$:

$$\omega = \sqrt{(2\pm\sqrt{2})\left(\frac{g}{L}\right)} , \tag{15}$$

$$\frac{\theta_{20}}{\theta_{10}} = \frac{-1\mp\sqrt{2}}{2\pm\sqrt{2}} . \tag{16}$$

The higher frequency mode has the masses moving out of phase, and the lower frequency mode has the masses moving in phase.

In the limit $\ell_2 \to 0$, one of the frequencies becomes infinite, corresponding to a simple pendulum of zero length. The other frequency can be evaluated using l'Hôpital's rule

$$\omega^2 = \frac{g}{\ell_1} , \tag{17}$$

which is the frequency of a simple pendulum of length ℓ_1, and

$$\frac{\theta_{20}}{\theta_{10}} = 1 , \tag{18}$$

corresponding to the two masses oscillating synchronously.

Hamilton's Equations

To derive the Hamiltonian equations for the double pendulum, start from the Lagrangian

$$L = T - V = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - m_1gy_1 - m_2gy_2 , \tag{19}$$

where T is the kinetic energy, and V is the potential energy, and g is the acceleration of gravity.

The generalized momenta can be found from the Lagrangian:

$$\begin{aligned} p_{\theta_1} = \frac{\partial L}{\partial\dot\theta_1} &= (m_1+m_2)\ell_1^2\dot\theta_1 + m_2\ell_1\ell_2\dot\theta_2\cos(\theta_1-\theta_2) , && (20)\\ p_{\theta_2} = \frac{\partial L}{\partial\dot\theta_2} &= m_2\ell_2^2\dot\theta_2 + m_2\ell_1\ell_2\dot\theta_1\cos(\theta_1-\theta_2) . && (21) \end{aligned}$$

The Hamiltonian is the Legendre transform of the Lagrangian:

$$H(\theta_1,\theta_2,p_{\theta_1},p_{\theta_2}) = \dot\theta_1 p_{\theta_1} + \dot\theta_2 p_{\theta_2} - L . \tag{22}$$

The velocities can be expressed as functions of the coordinates and momenta:

$$\dot\theta_1 = \frac{\ell_2 p_{\theta_1} - \ell_1 p_{\theta_2}\cos(\theta_1-\theta_2)}{\ell_1^2\ell_2\left[m_1 + m_2\sin^2(\theta_1-\theta_2)\right]} , \tag{23}$$

$$\dot\theta_2 = \frac{\ell_1(m_1+m_2)p_{\theta_2} - \ell_2 m_2 p_{\theta_1}\cos(\theta_1-\theta_2)}{\ell_1\ell_2^2 m_2\left[m_1 + m_2\sin^2(\theta_1-\theta_2)\right]} . \tag{24}$$

The Hamiltonian is

$$\begin{aligned} H \;=\; & \frac{m_2\ell_2^2 p_{\theta_1}^2 + (m_1+m_2)\ell_1^2 p_{\theta_2}^2 - 2m_2\ell_1\ell_2 p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{2\ell_1^2\ell_2^2 m_2\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]} \\ & -(m_1+m_2)g\ell_1\cos\theta_1 - m_2 g\ell_2\cos\theta_2 . \end{aligned} \tag{25}$$

Hamilton's equations for the time rate of change of the generalized momenta are

$$\dot p_{\theta_1} = -\frac{\partial H}{\partial\theta_1} = -\frac{\partial K}{\partial\theta_1} - (m_1+m_2)g\ell_1\sin(\theta_1) , \tag{26}$$

$$\dot p_{\theta_2} = -\frac{\partial H}{\partial\theta_2} = -\frac{\partial K}{\partial\theta_2} - m_2 g\ell_2\sin(\theta_2) , \tag{27}$$

where the generalized kinetic energy

$$K \equiv H - V = \frac{m_2\ell_2^2 p_{\theta_1}^2 + (m_1+m_2)\ell_1^2 p_{\theta_2}^2 - 2m_2\ell_1\ell_2 p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{2\ell_1^2\ell_2^2 m_2\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]} . \tag{28}$$

Note that K depends on the generalized coordinates only through the difference $\theta_1-\theta_2$. Hence

$$\begin{aligned} \frac{\partial K}{\partial\theta_2} = -\frac{\partial K}{\partial\theta_1} \;=\; & \frac{m_2\ell_2^2 p_{\theta_1}^2 + (m_1+m_2)\ell_1^2 p_{\theta_2}^2 - 2m_2\ell_1\ell_2 p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{2\ell_1^2\ell_2^2\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]^2}\sin[2(\theta_1-\theta_2)] \\ & -\frac{p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)}{\ell_1\ell_2\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]} \\ =\; & \frac{m_2 K\sin[2(\theta_1-\theta_2)]}{\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]} - \frac{p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)}{\ell_1\ell_2\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]} . \end{aligned} \tag{29}$$

Simulating the Double Pendulum

With Hamilton's equations in vector form, it is straightforward to apply the Runge-Kutta algorithm to solve for the trajectories of the double pendulum.

For the simulation, assume that the two masses are equal $m_1 = m_2 = m$ and that the two two lengths are also equal $\ell_1 = \ell_2 = L$. Hamilton's equations reduce to

$$\frac{d}{dt}\begin{pmatrix}\theta_1\\ \theta_2\\ p_{\theta_1}\\ p_{\theta_2}\end{pmatrix} = \begin{pmatrix} \frac{p_{\theta_1} - p_{\theta_2}\cos(\theta_1-\theta_2)}{mL^2[1+\sin^2(\theta_1-\theta_2)]} \\ \frac{2p_{\theta_2} - p_{\theta_1}\cos(\theta_1-\theta_2)}{mL^2[1+\sin^2(\theta_1-\theta_2)]} \\ -2mgL\sin\theta_1 - C_1 + C_2 \\ -mgL\sin\theta_2 + C_1 - C_2 \end{pmatrix} , \tag{30}$$

where

$$C_1 = \frac{p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)}{mL^2[1+\sin^2(\theta_1-\theta_2)]} , \tag{31}$$

$$C_2 = \frac{p_{\theta_1}^2 + 2p_{\theta_2}^2 - 2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)}{2mL^2[1+\sin^2(\theta_1-\theta_2)]^2}\sin[2(\theta_1-\theta_2)] . \tag{32}$$

Poincaré Section

Because the phase space of the system is 4-dimensional and hard to visualize, it is useful to look at a Poincaré section, for example, the intersections of the trajectory with a 2-dimensional slice of the phase space.

Because the Hamiltonian H is conserved in time, the trajectories actually lie on a 3-dimensional hypersurface determined by the equation

$$E = H + 3mgL = \frac{1}{2mL^2}\frac{p_{\theta_1}^2 + 2p_{\theta_2}^2 - 2p_{\theta_1}p_{\theta_2}\cos(\theta_1 - \theta_2)}{1 + \sin^2(\theta_1 - \theta_2)} + mgL(3 - 2\cos\theta_1 - \cos\theta_2)\ , \tag{33}$$

in the case of equal masses and lengths. A constant has been added to H so $E = 0$ at the stable equilibrium position.

A simple choice of section is obtained by focusing on the phase space of the upper mass, and recording its position and momentum θ_1, p_{θ_1} every time the trajectory crosses the hypersurface defined by $\theta_2 = 0$ when the second mass hangs momentarily vertically below the first. At this instant of time the lower mass can be moving in the clockwise or the counterclockwise directions: the Poincaré section can be simplified by plotting intersections only when the lower mass is moving clockwise, i.e., when $p_{\theta_2} > 0$.

C++ Program

The following program generates the pendulum trajectory and Poincaré section data.

Program 1: `http://www.physics.buffalo.edu/phy410-505/topic4/dpendulum.cpp`

```
#include <cmath>
#include <cstdlib>
#include <fstream>
#include <iostream>
#include <vector>
using namespace std;

#include "linalg.hpp"
#include "odeint.hpp"
using namespace cpl;

const double pi = 4 * atan(1.0);  // value of pi
const double g = 9.8;             // acceleration of gravity
double L = 1;                     // length of each pendulum
double m = 1;                     // mass of each pendulum bob

bool switch_t_and_theta_2 = false;// to zero in on section point

vector<double> f(                 // extended flow vector for Runge-Kutta
  const vector<double>& x         // extended solution vector
) {

  double t = x[0], theta_1 = x[1], theta_2 = x[2], p_1 = x[3], p_2 = x[4];

  double sin12 = sin(theta_1 - theta_2),
```

```
         cos12 = cos(theta_1 - theta_2),
         sin212 = sin(2 * (theta_1 - theta_2)),
         denom = 1 + sin12 * sin12;

  double C_1 = p_1 * p_2 * sin12 / (m * L * L * denom),
         C_2 = (p_1 *  p_1 + 2 * p_2 *  p_2 - 2 * p_1 * p_2 * cos12)
               * sin212 / (2 * m * L *  L * denom * denom);

  vector<double> f(5);
  f[0] = 1;
  f[1] = (p_1 - p_2 * cos12) / (m * L * L * denom);
  f[2] = (2 * p_2 - p_1 * cos12) / (m * L * L * denom);
  f[3] = - 2 * m * g * sin(theta_1) - C_1 + C_2;
  f[4] = - m * g * L * sin(theta_2) + C_1 - C_2;

  if (switch_t_and_theta_2) {
    double d_theta_2_dt = f[2];
    f /=  d_theta_2_dt;           // change variable from t to theta_2
  }

  return f;
}

double Energy(                    // conserved energy
  const vector<double>& x         // extended solution vector
) {

  double t = x[0], theta_1 = x[1], theta_2 = x[2], p_1 = x[3], p_2 = x[4];
  double sin12 = sin(theta_1 - theta_2),
         cos12 = cos(theta_1 - theta_2);

  return 1 / (2 * m * L * L * (1 + sin12 * sin12))
         * (p_1 * p_1 + 2 * p_2 *p_2 - 2 * p_1 * p_2 * cos(theta_1 - theta_2))
         + m * g * L * (3 - 2 * cos(theta_1) - cos(theta_2));
}

double p_theta_2(                 // solve for p_theta_2
  const double E,                 // given energy E
  const double theta_1,           // position of upper pendulum
  const double theta_2,           // position of lower pendulum
  const double p_1                // momentum of upper pendulum
) {

  double sin12 = sin(theta_1 - theta_2),
         cos12 = cos(theta_1 - theta_2),
         denom = 2 * m * L * L * (1 + sin12 * sin12);

  double a = 2 / denom,
         b = - 2 * p_1 * cos12 / denom,
         c = p_1 * p_1 / denom - E
             + m * g * L * (3 - 2 * cos(theta_1) - cos(theta_2));
```

```
  double discr = b * b - 4 * a * c;
  if (discr < 0) {
    cerr << " Sorry, E = " << E << " too small, cannot solve for p_theta_2"
         << endl;
    exit(EXIT_FAILURE);
  }

  return (- b + sqrt(discr)) / (2 * a);
}

int main() {

  cout << " Chaos in the Double Pendulum\n"
       << " ----------------------------\n"
       << " Enter E, theta_1, theta_2, p_theta_1: ";
  double E, theta_1, theta_2, p_1;
  cin >> E >> theta_1 >> theta_2 >> p_1;
  double p_2 = p_theta_2(E, theta_1, theta_2, p_1);
  cout << " Enter number of section points: ";
  int section_points;
  cin >> section_points;
  ofstream trajectory_file("trajectory.data");
  ofstream section_file("section.data");

  double t = 0;
  vector<double> x(5);            // extended solution vector
  x[0] = t; x[1] = theta_1; x[2] = theta_2; x[3] = p_1; x[4] = p_2;

  RK4 rk4;
  double dt = 0.001;
  double accuracy = 1e-6;
  rk4.set_step_size(dt);
  rk4.set_accuracy(accuracy);

  int crossing = 0;
  while (true) {

    trajectory_file << x[0] << ’\t’ << x[1] << ’\t’ << x[2]
                    << ’\t’ << x[3] << ’\t’ << x[4] << ’\n’;
    double theta_2_old = x[2];

    rk4.adaptive_step(f, x);

    if (theta_2_old < 0.0 && x[2] >= 0.0 && x[4] >= 0.0) {

      ++crossing;
      switch_t_and_theta_2 = true;
      double dt_save = rk4.get_step_size();

      vector<double> x_map = x;
```

```
      double d_theta_2 = -x_map[2];
      rk4.set_step_size(d_theta_2);
      rk4.step(f, x_map);

      switch_t_and_theta_2 = false;
      rk4.set_step_size(dt_save);
      cout << "Crossing " << crossing << " theta_1 = " << x_map[1]
           << " p_theta_1 = " << x_map[3] << " Energy = "
           << Energy(x_map) << endl;
      section_file << x_map[1] << ’\t’ << x_map[3] << ’\n’;
    }

    if (crossing >= section_points)
      break;
  }

  trajectory_file.close();
  section_file.close();
}
```

Fig. 2 shows two trajectories of the double pendulum with the same energy: one set of initial conditions results in a multiply periodic trajectory; a different set initial conditions lead to chaos!

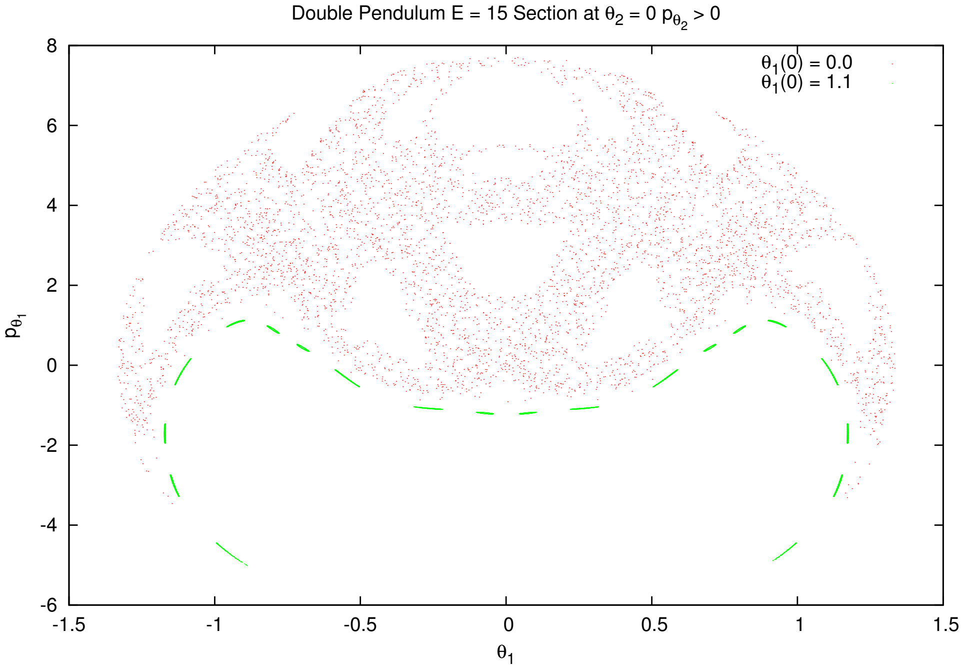

Figure 2: Poincaré sections with 5000 points for the double pendulum with $E = 15$. The chaotic trajectory (red dots) results from choosing initial conditions $\theta_1 = 0, \theta_2 = 0, p_{\theta_1} = 0$, and the periodic trajectory (green dots) from $\theta_1 = 1.1, \theta_2 = 0, p_{\theta_1} = 0$.

Homework Problem

First consider small oscillations. Simulate the two normal modes of the double pendulum: measure the normal mode frequencies and verify the mode behavior by plotting the trajectories $\theta_{1,2}(t)$.

Next, explore the full nonlinear behavior by generating Poincaré sections for energy $E = 1, 5, 10, 15, 40$. Try different initial conditions for each value of the energy E. For example, with $E = 15$, try the two different initial values sets (1) $\theta_1 = 1.1$, and $\theta_2 = 0$, and (2) $\theta_1 = 0$, and $\theta_2 = 0$. Describe qualitatively the types of motion you observe.

References

[1] T. Shinbrot, C. Grebogi and J. Wisdom, "Chaos in a double pendulum", Am. J. Phys. **60**, 491 (1992), `http://dx.doi.org/10.1119/1.16860`.

[2] R.B. Levien and S.M. Tan, "Double pendulum: An experiment in chaos", Am. J. Phys. **61**, 1038 (1993), `http://dx.doi.org/10.1119/1.17335`.

[3] Double pendulum Java applet from a previous course: `http://www.physics.buffalo.edu/gonsalves/phy410-505_fall00/Chapter6/oct27.html`.