

The Three-Body Problem

Gravitational N-body Problem

Consider N point masses m_i located at positions \mathbf{r}_i that interact with one another according to Newton's law of gravity. The energy of the system is

$$E = \frac{1}{2} \sum_i m_i \mathbf{v}_i^2 - G_N \sum_{i < j} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (1)$$

where G_N is the constant of universal gravitation, and

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} \quad (2)$$

is the velocity of m_i . The potential energy sum is over all pairs of masses. There are $N(N-1)/2$ pairs.

The N -body problem is to solve for the trajectories $\mathbf{r}_i(t)$ given initial values $\{\mathbf{r}_i(0), \mathbf{v}_i(0)\}$ at $t = 0$.

The $N = 3$ problem involves solving a system of 9 coupled second order differential equations:

$$\frac{d^2 \mathbf{r}_1}{dt^2} = -G_N m_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} - G_N m_3 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3}, \quad (3)$$

$$\frac{d^2 \mathbf{r}_2}{dt^2} = -G_N m_3 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3} - G_N m_1 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}, \quad (4)$$

$$\frac{d^2 \mathbf{r}_3}{dt^2} = -G_N m_1 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3} - G_N m_2 \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3}. \quad (5)$$

Search for Exact 3-Body Solutions

After Newton proved that the most general bounded solution of the classical equations of motion for two gravitating point masses is a Kepler ellipse, the search was on for exact solutions to the problem of three mutually gravitating point masses.

Euler found a special exact solution with the three particles moving in elliptical orbits in a collinear configuration in 1763. Lagrange found an exact planar configuration in 1772 with the three particles moving in elliptical orbits at the vertices of an equilateral triangle. Chenciner and Montgomery proved in 2000 that a numerical solution found by Moore with the three bodies chasing one another around a figure of eight path was exact. Chenciner's Scholarpedia article[2] has an authoritative description of the three-body problem. Several interesting "choreographies" were generated numerically by Simó[3] soon after the figure of eight solution was discovered.

Very few exact analytical results have been found in the three body problem. Poincaré showed that the problem is not integrable. Most orbits are unstable and may be chaotic, which means that arbitrarily close initial conditions result in orbits that separate exponentially and cannot be computed numerically for arbitrarily long times.

Symmetries and Integrals of Motion

The 3-body equations are invariant under translations in space and time, spatial rotations, and Galilean boosts. These symmetries imply that there are conservation laws, which can be used to simplify the problem.

Time translations: The equations are invariant under $t \rightarrow t + \tau$ where τ is a constant. The total energy

$$E = \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 + \frac{1}{2}m_3\mathbf{v}_3^2 - G_N \frac{m_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} - G_N \frac{m_2m_3}{|\mathbf{r}_2 - \mathbf{r}_3|} - G_N \frac{m_3m_1}{|\mathbf{r}_3 - \mathbf{r}_1|} \quad (6)$$

of the system is conserved. This gives one first integral of the motion.

Space translations: The equations are invariant under $\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{a}$ where \mathbf{a} is a constant vector. The total momentum of the system

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i \quad (7)$$

is conserved. This gives 3 first integrals of the motion.

Galilean boosts: The equations are invariant under $\mathbf{v}_i \rightarrow \mathbf{v}_i + \mathbf{u}$ where \mathbf{u} is a constant vector. The choice

$$\mathbf{u} = -\frac{\mathbf{P}}{\sum_i m_i} \quad (8)$$

transforms to the center-of-mass reference frame in which $\mathbf{P} = 0$. The origin of coordinates in this reference frame can be chosen so that

$$\mathbf{r}_{\text{cm}} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = 0. \quad (9)$$

Spatial rotations: The equations are invariant under $\mathbf{r}_i \rightarrow \mathbf{O} \mathbf{r}_i$ where \mathbf{O} is a constant 3×3 orthogonal matrix. The angular momentum

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i \quad (10)$$

is conserved. This gives 3 additional first integrals of motion.

Integrability

A system of N bodies with $6N$ degrees of freedom is said to be integrable if there exist $3N$ independent first integrals of motion. The solution can then be reduced to calculating definite integrals.

In the center-of-mass system, the 2-body Kepler problem has 3 independent conserved quantities: the energy, the angular momentum component perpendicular to the plane of the orbit, and the direction of the Runge-Lenz vector.

The 3-body problem has $6 \times 3 = 18$ degrees of freedom. The energy E , total momentum \mathbf{P} , and total angular momentum \mathbf{L} provide 7 independent first integrals of motion. The 3-body equations are not integrable!

Euler's Collinear Solution

Euler[1] found

Lagrange's Equilateral Triangle Solution

Chenciner-Montgomery Solution

C++ Code for the Three-Body Problem

_____ Program 1: <http://www.physics.buffalo.edu/phy410-505/topic3/three-body.cpp> _____

```

// ***** UNDER CONSTRUCTION - DO NOT USE *****

#include "cstdlib"
#include "iostream"
#include "vector"
using namespace std;

#include "vector.hpp"
#include "odeint.hpp"

using namespace cpl;

vector<double> m(3);           // masses of three bodies

vector< vector<double> >    // vector of vectors, note space in "> >" !!
  r(3, vector<double>(2)),   // three (x, y) position vectors
  v(3, vector<double>(2));   // three (v_x, v_y) velocity vectors

enum solution {
  EULER, LAGRANGE, CHENCINER_MONTGOMERY
};

void initialize(solution s) {

  for (int i = 0; i < 3; i++)
    m[i] = 1;

}

int main() {

  cerr << "This program is not complete - do not use" << endl;
  exit(EXIT_FAILURE);

}

```

Homework Problem

Explore Simó's choreographies, choose one which you find interesting, and try to reproduce it with the program. Study the stability of this solution, and the Euler, Lagrange, and Chenciner-Montgomery solutions. You can do this by making small changes to the initial values of the variables to see whether the bound state orbits retain their characteristic shape or whether the system falls apart. Which of these solutions are stable, and which are not?

References

- [1] L. Euler, “De motu rectilineo trium corporum se mutuo attrahentium”, *Novi Commentarii academiae scientiarum Petropolitanae* **11**, 144 (1767), <http://math.dartmouth.edu/~euler/pages/E327.html>.
- [2] A. Chenciner, “Three body problem”, http://www.scholarpedia.org/article/Three_body_problem.
- [3] C. Simó, “Dynamical properties of the figure eight solution of the three-body problem”, <http://www.maia.ub.es/dsg/2001/0104simo.zip>; “Choreographies of the Planar Three Body Problem”, <http://www.maia.ub.es/dsg/3body.html>.
- [4] E. Butikov, “Collection of Remarkable Three-Body Motions”, <http://faculty.ifmo.ru/butikov/Projects/Collection.html>, Java Applets and explanations.