Chapter 15: Superposition and Interference of Waves

“Real” waves are rarely purely sinusoidal (harmonic), but they can be represented by superpositions of harmonic waves

In this chapter we explore what happens when harmonic waves combine to form more complicated waves
The solutions to the wave equation are linear

\[ \text{True for small displacements where the restoring force remains linear} \]

What does this mean?

If \( a(x, t) \) and \( b(x, t) \) are solutions to the wave equation, then so is

\[ z(x, t) = a(x, t) + b(x, t) \]

Check:

\[
\frac{\partial^2 z}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2} \quad \Rightarrow \quad \frac{\partial^2 (a+b)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 (a+b)}{\partial t^2}
\]

\[
\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 b}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 a}{\partial t^2} + \frac{1}{v^2} \frac{\partial^2 b}{\partial t^2}
\]

is true if

\[
\frac{\partial^2 a}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 a}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 b}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 b}{\partial t^2}
\]

The sum (or difference) of two solutions of the wave equation is again a solution of the wave equation

\[ \Rightarrow \text{superposition principle} \]
**Interference**

The superposition principle allows us to add waves of different amplitudes, wavelengths, and frequencies, each of them moving in different directions.

=> The result: **Interference**

Mathematically: the sums are *algebraic* sums

Examples:

(Red curves: algebraic sum of the blue and green curves)
Important trigonometric relationships for general cases of interference:

\[
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)
\]

\[
\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)
\]

Simple example for **destructive** interference:

\[
z_1(x, t) = z_0 \sin(kx - \omega t)
\]

\[
z_2(x, t) = z_0 \sin(kx - \omega t + \pi) = -z_0 \sin(kx - \omega t)
\]

\[
\Rightarrow z_1 + z_2 = 0
\]

Simple example for **constructive** interference:

\[
z_1(x, t) = z_0 \sin(kx - \omega t)
\]

\[
z_2(x, t) = z_0 \sin(kx - \omega t)
\]

\[
\Rightarrow z_1 + z_2 = 2z_0 \sin(kx - \omega t)
\]

Important trigonometric relationships for general cases of interference:

\[
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)
\]

\[
\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b)
\]

Example:

\[
z_1(x, t) = z_0 \sin(kx - \omega t)
\]

\[
z_2(x, t) = z_0 \sin(kx - \omega t + \phi)
\]

\[
\Rightarrow z_1 + z_2 = ?
\]

\[
z_1 + z_2 = z_0[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]
\]

\[
= z_0[\sin(a) + \sin(a + b)]
\]
use: 
\[ \sin(kx - \omega t + \varphi) = \sin(kx - \omega t) \cos(\varphi) + \cos(kx - \omega t) \sin(\varphi) \]

plug in:
\[ z_1 + z_2 = z_0 [\sin(kx - \omega t) + \sin(kx - \omega t + \varphi)] \]
\[ = z_0 [\sin(kx - \omega t) + \sin(kx - \omega t) \cos(\varphi) + \cos(kx - \omega t) \sin(\varphi)] \]
\[ = z_0 (1 + \cos(\varphi)) \sin(kx - \omega t) + \cos(kx - \omega t) \sin(\varphi) \]

=> if \( \varphi = 0 \rightarrow z_1 + z_2 = 2z_0 \sin(kx - \omega t) \)
if \( \varphi = \pi \rightarrow z_1 + z_2 = 0 \)

**Coherence:** Phase is stable in time.
Standing Waves Through Interference

Adding two oppositely traveling waves to get a standing wave

\[ z_{\text{left}}(x, t) = z_0 \sin(\omega t + kx) \]
\[ z_{\text{right}}(x, t) = z_0 \sin(\omega t - kx) \]

\[ z(x, t) = z_{\text{right}}(x, t) + z_{\text{left}}(x, t) \]
\[ z(x, t) = z_0 \left[ \sin(\omega t - kx) + \sin(\omega t + kx) \right] \]
\[ = z_0 \left[ \sin(\omega t) \cos(-kx) + \sin(-kx) \cos(\omega t) + \right] \]
\[ \sin(\omega t) \cos(+kx) + \sin(+kx) \cos(\omega t) \]
\[ = 2z_0 \sin(\omega t) \cos(kx) \]

\[ z(x, t) = 2z_0 \sin(\omega t) \cos(kx) \]

note that \( \omega \) and \( k \) are same for traveling waves and resulting standing wave
Superposition of traveling waves
Interference of incident and reflected waves

To get node at wall, reflected wave must have opposite sign as incident wave (but same magnitude at wall)
Beats occur when two waves of nearly the same frequency are superposed.

Example: Two combs with different tooth spacing

Comb with 64 teeth and 20 cm long => \( \lambda = 0.3125 \) cm

Comb with 64 teeth and 21 cm long => \( \lambda = 0.3281 \) cm

Distance between “pulses” \( \sim 6.8 \) cm

Where the two combs superpose, a new pattern emerges (here called Moire pattern).

A similar thing happens when two waves superpose.
Two sine waves with nearly same wavelengths superpose to a wave that has an envelope that is again a sine wave but with longer wavelength. => beats

How to calculate the wavelength of the beats?
To simplify analysis, fix position \( x \) and look at wave as function of time. (Could equivalently fix time \( t \) and look at wave as a function of position)

\[
z_1(x, t) = z_0 \sin(\omega_1 t - kx) \rightarrow z_0 \sin(\omega_1 t)
\]

\[
z_2(x, t) = z_0 \sin(\omega_2 t - kx) \rightarrow z_0 \sin(\omega_2 t)
\]

Lets define two new frequencies:

\[
\Omega = \frac{1}{2} (\omega_1 + \omega_2) \quad \text{and} \quad \delta\omega = \omega_1 - \omega_2
\]

\[
z_1(x, t) = z_0 \sin \left[ \left( \frac{\Omega + \delta\omega}{2} \right) t \right] = z_0 \sin(\Omega t) \cos \left( \frac{\delta\omega}{2} t \right) + z_0 \cos(\Omega t) \sin \left( \frac{\delta\omega}{2} t \right)
\]

\[
z_2(x, t) = z_0 \sin \left[ \left( \frac{\Omega - \delta\omega}{2} \right) t \right] = z_0 \sin(\Omega t) \cos \left( \frac{-\delta\omega}{2} t \right) + z_0 \cos(\Omega t) \sin \left( \frac{-\delta\omega}{2} t \right)
\]

\[
z_1(x, t) + z_2(x, t) = 2z_0 \sin(\Omega t) \cos \left( \frac{\delta\omega}{2} t \right)
\]

average frequency: \( \Omega \) (fast)

beat frequency: \( \omega_{\text{beat}} = \frac{1}{2} \delta\omega \) (slow)
Note: since the ear is only sensitive to the amplitude, it cannot tell the difference between

$$\cos\left(\frac{1}{2}\delta \omega t\right) \quad \text{and} \quad -\cos\left(\frac{1}{2}\delta \omega t\right)$$

$$\Rightarrow$$ the **perceived** beating frequency is

$$f_{\text{pulse}} = 2 \cdot f_{\text{beat}} = f_1 - f_2 = \text{pulse frequency}$$
Demo: Beat frequency

\[ \bar{f} = \frac{\Omega}{2\pi} \] (fast)

And pulse frequency

\[ f_{\text{pulse}} = 2 \cdot f_{\text{beat}} = f_1 - f_2 \]

What happens to the pulse frequency when the two frequencies get close to each other?

What happens to the pulse frequency when the two frequencies move apart?
Spatial Interference Phenomena

(fix time t by taking snapshot and look at x-dependence)
In phase:
Constructive interference

\[ \Delta x = n\lambda \]
\[ n = 0, \pm 1, \pm 2, \ldots \]

Phase difference:
\[ n \cdot 2\pi \]

Out of phase:
Destructive Interference

\[ \Delta x = (n + \frac{1}{2})\lambda \]
\[ n = 0, \pm 1, \pm 2, \ldots \]

Phase difference:
\[ (n + \frac{1}{2}) \cdot 2\pi \]
\[ \Delta L = L_2 - L_1 \]

minima when \( \Delta L = (n + 1/2) \lambda \),

maxima when \( \Delta L = n \lambda \), \( n = 0, \pm 1, \pm 2, \ldots \)

Recall:

\[ \lambda \rightarrow 2\pi \quad \cos(x + \lambda) = \cos(x) \]

\[ \frac{\lambda}{2} \rightarrow \pi \quad \cos(x + \frac{\lambda}{2}) = -\cos(x) \]

\[ \frac{\lambda}{4} \rightarrow \frac{\pi}{2} \quad \cos(x + \frac{\lambda}{4}) = -\sin(x) \]
Locations of maxima and minima:
\[ \Delta L = d \sin \theta \]

for maxima:
\[ \Delta L = n\lambda \quad \rightarrow \quad \sin \theta = \frac{n\lambda}{d} \]

for minima:
\[ \Delta L = \left( n + \frac{1}{2} \right) \lambda \quad \rightarrow \quad \sin \theta = \frac{(n + \frac{1}{2})\lambda}{d} \]

where \( n = 0, \pm 1, \pm 2, \pm 3 \ldots \)
Demo: Interference from two speakers

Raise your hand when an acoustic minimum passes you.
Collision of pulses
Note that the center point at the dot remains motionless.

⇒ Connection to reflection of pulse on fixed end
Reflection from a fixed end

Think of this as a “real” pulse moving to the right and an “imaginary” pulse moving to the left.

**Before reflection:** only “real” pulse visible (“imaginary” one is somewhere beyond wall)

**After reflection:** previously “imaginary” pulse is real now and moves to left

What happens **during** the reflection?
⇒ superposition of left and right moving pulses
  (such that rope end remains fixed, see previous slide)
Reflection from a vertically free end

What happens here?
Reflection due to change in $\mu$ (note tension is constant!)

$v \propto ?$

(a) Lighter to heavier medium
(b) heavier to lighter medium
Power is conserved (power in=power out)

\[
\frac{\mu_1 \omega_i^2 A_i^2 v_i}{2} = \frac{\mu_1 \omega_r^2 A_r^2 v_r}{2} + \frac{\mu_2 \omega_t^2 A_t^2 v_t}{2}
\]

\[v_i = v_r = v_1 \text{ and } v_t = v_2\]

\[\omega_i = \omega_r = \omega_t\]
Fourier decomposition of waves
This is analogous to decomposing a vector into orthogonal direction components \((x, y, \text{ and } z)\), except here we are decomposing a function into orthogonal frequency components.
\[ z(t) = \sum_{n=\text{odd}}^{\infty} \frac{1}{\pi} \frac{1}{n^2} \cos(n\omega t) \]

\[ = (0.81)z_0 \left[ \cos(\omega t) + \frac{1}{9} \cos(3\omega t) + \frac{1}{25} \cos(5\omega t) + \ldots \right] \]
Real Waves

(a) Trumpet

(b) Synthesized trumpet

(c) Guitar

(d) Synthesized guitar
Ch 15 Useful Equations

$$f_{\text{beat}} = \frac{1}{2} (f_1 - f_2)$$

Interference due to two sources

$$\Delta L = d \sin \theta$$

if two sources are in phase ($\Delta \phi = 0$):

for maxima: $\Delta L = n \lambda \quad \rightarrow \quad \sin \theta = \frac{n \lambda}{d}$

for minima: $\Delta L = \left( n + \frac{1}{2} \right) \lambda \quad \rightarrow \quad \sin \theta = \frac{\left( n + \frac{1}{2} \right) \lambda}{d}$

where $n = 0, \pm 1, \pm 2, \pm 3 \ldots$

If two sources are out of phase ($\Delta \phi = \pi$)

for maxima: $\Delta L = \left( n + \frac{1}{2} \right) \lambda \quad \rightarrow \quad \sin \theta = \frac{\left( n + \frac{1}{2} \right) \lambda}{d}$

for minima: $\Delta L = n \lambda \quad \rightarrow \quad \sin \theta = \frac{n \lambda}{d}$

$$\frac{\mu_i \omega_i^2 A_i^2 v_i}{2} = \frac{\mu_r \omega_r^2 A_r^2 v_r}{2} + \frac{\mu_t \omega_t^2 A_t^2 v_t}{2}$$

$v_i = v_r = v_1$ and $v_t = v_2 \quad \omega_i = \omega_r = \omega_t$