Maximizing the range of the shot put

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We address the problem of the optimum angle at which a putter should release the shot in order to achieve maximum distance.

I. OPTIMAL ANGLE NEGLECTING AIR RESISTANCE

In most elementary physics textbooks, it is stated that the maximum range of a projectile is attained when the angle of firing is 45°. However, film studies by Cureton[1] of shot putters in action have shown that the best putters release the shot at an angle of between 40° and 42° from the horizontal. We have spoken to several people, both in physics and in coaching, and they seemed genuinely puzzled by the discrepancy. (A common reaction among physicists was to attribute the difference to the effect of air resistance, but as we shall show in Sec. II, air resistance does not appreciably affect the optimum angle.)

The most recent film study of shot putters was carried out by Dessureault.[2] In filming 13 shot putters, from novices to world record holders, he found that the angle of release of the best put was 43° and that the mean angle was 38°. Dessureault gave no explanation for these angles, stating that "biomechanics is still in the measurement period," and quoting Booysen[3] to the effect that "only by measuring of thousands of the world's best in every track and field event can the sought-for definite standard, an objective possibility of comparison, be found."

On the contrary, it is easy to show by using elementary mechanics that the best angle of release of the shot put is around 42°. The reason that the angle is not 45° is that the shot is released at a finite (nonzero) height above the ground. We assume that a putter releases the shot with initial speed v at a height h above the ground making an angle θ with the horizontal. We treat h, v, and θ as independent variables, although in actual practice, the putter may have to compromise in achieving the highest possible values of h and v, and optimum value of θ. The horizontal distance or range the shot goes before striking the ground we call R. The quantities h, v, θ, and R are shown in the diagram of Fig. 1.

For simplicity, we first treat the problem neglecting air resistance. The horizontal distance x the shot travels during a time t is

$$x = vt\cos\theta. \quad (1)$$

The shot is subject to the acceleration of gravity g. Its vertical position z at any instant is

$$z = h + vt\sin\theta - (\tfrac{1}{2})gt^2. \quad (2)$$

The shot hits the ground at a time T, when $z = 0$. At the time T, x is equal to the range R. Then, from Eqs. (1) and (2), we get

$$R = vT\cos\theta \quad (3)$$

$$0 = h + vT\sin\theta - (\tfrac{1}{2})gT^2. \quad (4)$$

We solve Eq. (3) for T and substitute into Eq. (4) to eliminate T. We obtain

$$0 = h + R\tan\theta - (\tfrac{1}{2})gR^2\sec^2\theta/v^2. \quad (5)$$

Solving for R, we get

$$R = v^2\cos\theta[\sin\theta + (\sin^2\theta + 2gh/v^2)^{1/2}]/g. \quad (6)$$

By inspection of Eq. (6), we see that increasing h and v will increase R. Thus, as expected, the putter should strive to maximize both the speed and height of release.

To find the maximum range as a function of θ, we must take the derivative of R with respect to θ and set it equal to zero. We obtain

$$\sin^2\theta_m = (2 + 2gh/v^2)^{-1}, \quad (7)$$

where θ_m is the angle which gives the maximum range R_m. From Eq. (7), if we know the height at which the shot leaves the hand and the speed of release, we can calculate the angle which will give the maximum range. We can use that angle in Eq. (6) to find the range. Usually, however, the initial speed v is not known and may be inconvenient to measure. We can eliminate v by solving Eq. (7) for v and substituting into Eq. (6) with $R = R_m$, $\theta = \theta_m$. We obtain the surprisingly simple result

$$R_m = h\tan 2\theta_m. \quad (8)$$

Equation (8) is a convenient one to obtain the best angle. As an example, suppose the shot leaves the putter's hand at a height of 7 ft above the ground. (This is about the average height of release.) Also suppose that the putter wants to set a world record of about 73 ft. Substituting these numbers into Eq. (8), we find

$$\theta_m = 42.3°.$$

It can be easily seen from Eq. (8) that for a weaker shot putter, the angle should be somewhat less. However, the best angle does not change very much. For example, for a put of 50 ft with a height of release of 7 ft, the best angle is 41°; for a put of 35 ft, the optimum angle is 39°.

We can use Eqs. (6) and (7) to obtain the maximum range in terms of the initial speed and height. We get

$$R_m = v^2(1 + 2gh/v^2)^{1/2}/g. \quad (9)$$

We note that if $h = 0$, Eqs. (8) and (9) reduce to the elementary textbook formulas $\theta_m = 45°$, $R_m = v^2/g$, as they should.

We can obtain another useful equation by eliminating h between Eqs. (8) and (9). We get

$$R_m = v^2\cot\theta_m/g. \quad (10)$$

Although we are primarily interested in the effect of θ on R, we briefly comment on the effects of h and v. Because the optimum angle is near 45° and because h is small compared to R, an increase in height Δh increases R by

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about the same amount. Similarly, R goes approximately as v^2, so that a 1% increase in the speed of release increases R by almost 2%. Both of these results can be easily seen by expanding Eq. (9), assuming $v^2 \gg gh$, which is equivalent to assuming $R_m \gg h$. We get

$$R_m \cong v^2/g + h. \tag{11}$$

It is interesting to write R_m in terms of the kinetic energy $K = (\frac{1}{2})Mv^2$ of the shot at release and the potential energy $V = Mgh$, where M is the mass of the shot. Then Eq. (11) becomes

$$R_m \cong (2K + V)/(Mg). \tag{12}$$

Thus, if the putter can give the shot a certain amount of energy, he is twice as well off by giving the shot kinetic rather than potential energy. But, of course, the putter cannot freely trade off between these two types of energy. As an extreme example, if the putter released the shot at ground level, his motion would be so awkward that he would be able to give the shot only very little kinetic energy.

To increase either v or h, the putter must impart additional energy to the shot. This, in general, costs him additional energy. On the other hand, by releasing the shot closer to the optimum angle, the putter increases the range without delivering additional energy to the shot. To a first approximation, this latter option should not cost him additional energy.

Because the range is stationary at the optimum angle, small errors in θ do not appreciably affect R. On a put of about 70 ft, an error of 1° shortens R by about ½ in.,[4] and an error of 3° shortens R by about 4 in. As the angle deviates more and more from the optimum, the effect on R gets progressively worse. For example, a 10° error in θ shortens R by about 3½ ft.

We next treat the problem including air resistance. We find that although the range of a 70-ft put is reduced by about ½ ft in still air, the optimum angle is changed by about 0.13°, which is a negligible amount. (A 0.13° error in θ changes R by about 0.01 in.)

II. EFFECTS OF AIR RESISTANCE

In this section, we shall show that air resistance shortens the range of a 70-ft put by about ½ ft, but has a negligible effect on the optimal angle. If we include air resistance, additional variables enter the problem: namely, the angular velocity of the shot and the axis of rotation. However, we shall not consider these variables, assuming that for a 16-lb shot, any slight rotation imparted by the putter has a negligible effect on the range.

The equation of motion of the shot is

$$M\, d\mathbf{u}/dt = -M\mathbf{g} - D(\mathbf{u} - \mathbf{w})/|\mathbf{u} - \mathbf{w}|, \tag{13}$$

where D is the drag arising from the air, $\mathbf{u}$ is the instantaneous velocity of the shot, and $\mathbf{w}$ is the wind velocity. The drag of an object can be written

$$D = (\tfrac{1}{2})cA\rho(\mathbf{u} - \mathbf{w})^2, \tag{14}$$

where A is the cross-sectional area of the object in the direction of motion, ρ is the density of the resistive medium, and c is a dimensionless quantity called the drag coefficient. The drag coefficient is a complicated function of the shape of the object and of the Reynolds number, which in turn, is proportional to the velocity. However, for a sphere the size

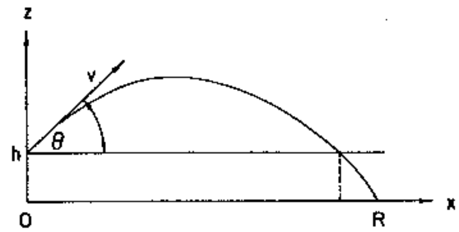

Fig. 1. Kinematic variables involved in the shot put.

of the shot traveling at the speeds that putters normally achieve, c is approximately a constant[5] with the value $c \approx \frac{1}{2}$.

Substituting Eq. (14) into Eq. (13) and writing the equation in component form we obtain

$$du_x/dt = -(\tfrac{1}{2})cA\rho|\mathbf{u} - \mathbf{w}|(u_x - w_x)/M, \tag{15}$$

$$du_y/dt = -(\tfrac{1}{2})cA\rho|\mathbf{u} - \mathbf{w}|(u_y - w_y)/M, \tag{16}$$

$$du_z/dt = -g - (\tfrac{1}{2})cA\rho|\mathbf{u} - \mathbf{w}|(u_z - w_z)/M, \tag{17}$$

where the y direction is perpendicular to the initial velocity $\mathbf{v}$. Equations (15)–(17) are coupled differential equations which cannot be integrated analytically in terms of elementary functions.[6] We have solved Eqs. (15)–(17) numerically in the case of still air. However, rather than present the details of our numerical solution, we find it more instructive to give an approximate analytic solution, since the effects of the drag are small in any case.

We begin our approximate treatment by writing

$$|\mathbf{u} - \mathbf{w}| \simeq [(u_x - w_x)^2 + w_y^2 + u_z^2 + w_z^2]^{1/2}. \tag{18}$$

We have dropped the term u_y because it is very small (being perpendicular to the direction of v), and have dropped the term $-2u_zw_z$ because u_z is in the same direction as w_z for about half the trajectory and opposite to w_z for the other half. We are, of course, assuming that $\mathbf{w}$ remains constant while the shot is in the air. We next replace our approximate expression for $|\mathbf{u} - \mathbf{w}|$ by some appropriate constant value γ. We can get upper and lower limits on the effect of the drag. We get an upper limit on γ by replacing the components of $\mathbf{u}$ by the components of $\mathbf{v}$ in Eq. (18), as u exceeds v only near the end of the trajectory. We get a lower limit by taking only u_x different from 0 and replacing u_x by $v_x = v\cos\theta$, as $v\cos\theta$ is approximately the minimum speed of the shot. Putting in these limits, we obtain

$$v^2\cos^2\theta + w^2 - 2vw_x\cos\theta < \gamma^2 < v^2 + w^2 - 2vw_x\cos\theta. \tag{19}$$

Thus we will be able to bracket the shortening of the range, and the average of our upper and lower limits will be pretty close to the actual value.

With these approximations, Eqs. (15)–(17) become

$$du_x/dt = -\alpha(u_x - w_x), \tag{20}$$

$$du_y/dt = -\alpha(u_y - w_y), \tag{21}$$

$$du_z/dt = -g - \alpha(u_z - w_z), \tag{22}$$

where

$$\alpha = (\tfrac{1}{2})cA\rho\gamma/M. \tag{23}$$

Equations (20)–(22) may be readily integrated in terms of elementary functions. Putting in the initial conditions, we obtain

$$x = (v \cos\theta - w_x)(1 - e^{-\alpha t})/\alpha + w_x t, \quad (24)$$

$$y = -w_y(1 - e^{-\alpha t})/\alpha + w_y t, \quad (25)$$

$$z = h + (v \sin\theta + g/\alpha - w_z)(1 - e^{-\alpha t})/\alpha + (w_z - g/\alpha)t. \quad (26)$$

As before, it is possible to set $x = R$ and $z = 0$ when $t = T$, and to substitute the expression for T from Eq. (24) into Eq. (26). However, the resulting expression is rather unwieldy. It is better to remember that α is small and to expand Eqs. (24) and (26) to first order in α.

We get

$$x = vt \cos\theta + (\tfrac{1}{2})\alpha t^2(w_x - v \cos\theta), \quad (27)$$

$$z = h + vt \sin\theta - (\tfrac{1}{2})gt^2 + (\tfrac{1}{2})\alpha t^2[w_z - v \sin\theta + (\tfrac{1}{3})gt]. \quad (28)$$

We now set $x = R$, $z = 0$ when $t = T$. Then Eq. (27) becomes

$$T = R \sec\theta/v + (\tfrac{1}{2})\alpha T^2(1 - w_x \sec\theta/v). \quad (29)$$

Since we are interested in T only to order α, we can substitute $T = R \sec\theta/v$ in the second term on the right-hand side of Eq. (29). We substitute the resulting expression into Eq. (28) with $z = 0$, and $t = T$. We obtain to order α

$$0 = h + R \tan\theta - (\tfrac{1}{2})gR^2 \sec^2\theta/v^2 + \alpha R^2 \sec^2\theta[(\tfrac{1}{2})gw_x R \sec^2\theta/v^2 - (\tfrac{1}{2})w_x \tan\theta + (\tfrac{1}{2})w_z - (\tfrac{1}{3})gR \sec\theta/v]/v^2. \quad (30)$$

We are interested in the change in range when α is small. We get this quantity by taking the derivative of R with respect to α and evaluating when $\alpha = 0$. We obtain the following rather complicated expression

$$dR/d\alpha = R^2(2gRv \cos\theta + 3v^2 w_x \sin\theta \cos\theta - 3gw_x R - 3v^2 w_z \cos^2\theta)/6v^2 \cos^2\theta(v^2 \cos\theta \sin\theta - gR). \quad (31)$$

We are interested in change in R at the optimum angle. In this case, R is given approximately by Eq. (10). Substituting Eq. (10) into Eq. (31), and letting $d\alpha = \alpha$, we get the simpler expression

$$dR_m = \frac{-\alpha v^3}{g^2 \sin\theta_m}\left(\frac{1}{3 \sin\theta_m \cos\theta_m} - \frac{w_x}{2v \sin\theta_m} - \frac{w_z}{2v \cos\theta_m}\right), \quad (32)$$

where α is given by Eq. (23).

Let us evaluate Eq. (32) in the case of still air. If we take

$$v = 46 \text{ ft/s}, \quad \theta_m = 42°, \quad A = 0.12 \text{ ft}^2$$
$$c = 0.5, \quad \rho = 0.071 \text{ lb/ft}^3, \quad M = 16 \text{ lb} \quad (33)$$

with $v \cos\theta_m < \gamma < v$, we find

$$-7 < dR_m < -5 \text{ in.} \quad (34)$$

for a put of about 73 ft. Thus air resistance shortens the range by about ½ ft. This value agrees very well with the results of our numerical calculation which gives $dR_m = -5.9$ in. For shorter puts, the effect of air resistance is considerably less, since as we can see from Eq. (32), dR_m goes approximately like v^4 (remember that α goes approximately like v) or like R_m^2.

Because the correct value for γ in the case of still air is so close to halfway between the limits, let us approximate γ^2 by taking the mean of the upper and lower limits of γ^2. This gives

$$\gamma^2 = (\tfrac{1}{2})v^2(1 + \cos^2\theta) + w^2 - 2vw_x \cos\theta. \quad (35)$$

Using this expression for γ, we evaluate the change in range as an example if there is a wind of speed half the speed of the shot or about 23 ft/s. We keep the conditions of Eq. (33). We find that the range is shortened a little less than 2 in. for a tail wind, about 14 in. for a head wind, and about 7 in. for a cross wind. These values of course should be compared with a shortening of 6 in. in still air. It is amusing that a cross wind shortens the range compared to the range in still air.

We next consider the change in the optimum angle arising from air resistance. It suffices to consider the case of still air. We set $\mathbf{w} = 0$ in Eq. (30), take the derivative of R with respect to θ, and set it equal to zero. This gives us the condition

$$R_m = v^2 \cot\theta_m(1 - \alpha v \csc\theta_m/g)/g. \quad (36)$$

Substituting Eq. (36) back into Eq. (30) to eliminate R_m, we obtain an expression for θ_m to first order in α. Taking the derivative of θ_m with respect to α in this expression, evaluating it at $\alpha = 0$, and again setting $d\alpha = \alpha$, we get

$$d\theta_m = -\alpha v(\cos\theta_m - \sec\theta_m/3)/g, \quad (37)$$

where $d\theta_m$ is expressed in radians. Putting in the numbers of Eq. (33) and changing to degrees, we find

$$d\theta_m = -0.13°. \quad (38)$$

As we have remarked, this change in the optimum angle has a negligible effect on the range. This can be verified directly by evaluating Eq. (6) with the optimum angle and with the angle changed by 0.13°. It would be wrong to use Eq. (8) or (10) to find the change in R arising from a small change in θ, because these equations are correct only at $\theta = \theta_m$, and, unlike Eq. (6), are not stationary with respect to small changes from the optimum.

According to the rules governing the shot put, the diameter of the shot may not be less than 4.331 in. nor more than 5.118 in. There is a 40% difference in area between the two extremes. We see from Eq. (32) that the loss in range is directly proportional to α, which, in turn, is directly proportional to the cross sectional area A. In computing a loss of range $dR_m = -\tfrac{1}{2}$ ft, we used a value of A about midway between the two extremes. The actual value of dR_m can vary by ±20% just because of the size of the shot. Thus, a world-class putter who uses the smallest permissible shot will have a little more than a 2-in. advantage over a putter who uses the largest allowed shot, all other things being equal.

In all our computations, we used a nominal value of $g = 32$ ft/s^2. Actually, g varies from place to place on the earth's surface. Similarly, the density of air depends on weather and altitude. All these variables together can change the range of a put by several inches, but our conclusions about the optimum angle remain valid.

In all our considerations we have been concerned with the path of the shot from the moment of release until the time of landing. We have not considered how, in fact, the putter achieves the maximum v and h and the optimum θ. The principles of mechanics can help here too, but the problem is much more difficult. At present, a large part of our understanding of this part of the problem must come, as Dessureault and Booysen believe, from observing the best shot putters in action.

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$$a = W/m = (W/mg)g$$

to the shot (neglecting the weight of the shot). If his arm has length l the final velocity will be

$$v = [2gl(W/mg)]^{1/2}$$

using $W = 150$ lb, $l = 3$ ft, $mg = 16$ lb, we find

$$v \simeq 42 \text{ ft/s}.$$

Since the putter imparts additional velocity from his initial movement in the ring this would probably account for the required 47 ft/s. Although this estimate is rough it shows directly the basic strength required to set a world record."

[1]T. K. Cureton, "Elementary Principles and Techniques of Cinematographic Analysis," Res. Quarterly **10**, 3 (1939).

[2]Jacques Dessureault, "Selected Kinetic and Kinematic Factors Involved in Shot Putting," Ph.D. thesis (Indiana University, 1976).

[3]Hannes Booysen, "Thoughts on Shot Put Technique," Track Technique **43**, 1365 (1971).

[4]Distances in the shot put are measured to the nearest ¼ inch.

[5]L. Prandtl and O. G. Tietjens, *Applied Hydro- and Aerodynamics*, (McGraw-Hill, New York, 1934), p. 100.

[6]J. M. J. Kooy and J. W. H. Uytenbogaart, *Ballistics of the Future* (Technical Publishing Co. H. Stam, Haarlem, Holland, 1946), p. 123.